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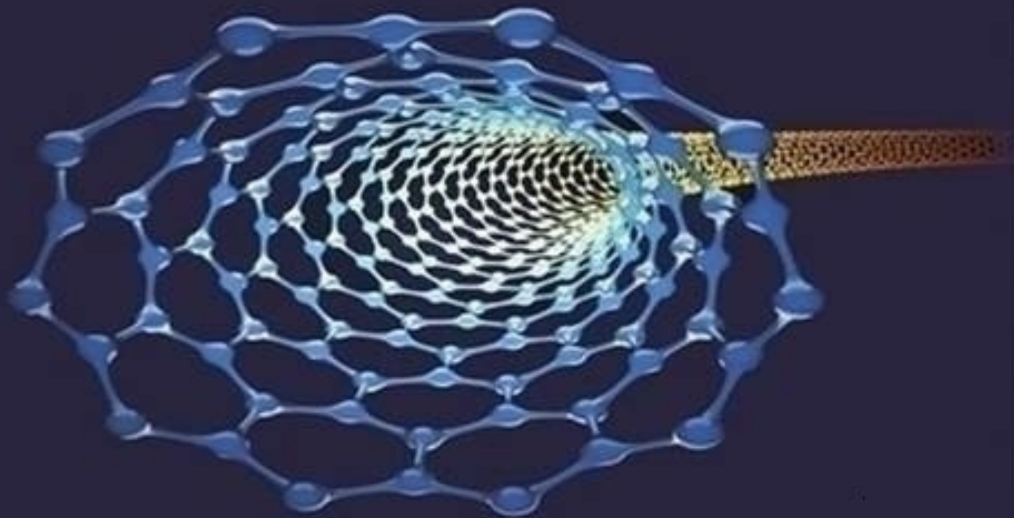

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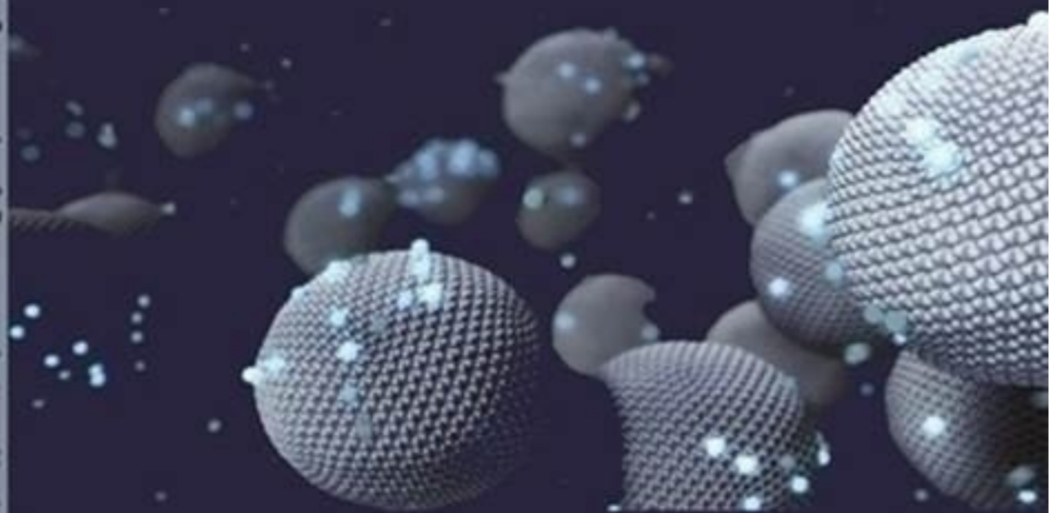

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RESEARCH PAPER

Purely Semismall Compressible Modules

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A B S T R A C T:
Let R be a ring with 1 and D be unitary left Module over R. In this paper, we present purely semi small compressible Modules. Also, we give remarks and examples, many properties of such Modules are investigated.

KEY WORDS: Semi small Submodule, prime Modules, pure Modules, semi small compressible Module.
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1. INTRODUCTION:

A Submodule W of an R-Module D is small Submodule ($W \ll D$) if $W + U = D$ for any Submodule U of D then $U = D$[1]. A proper Submodule W of an R-Module D is semismall of D ($W \ll_S D$) if W = 0 or $W/U \ll D/U$ $\forall$ nonzero Submodule U of W [2]. An R-Module D is semismall compressible (S.S.C.) if D embedded in all its non-zero semismall Submodule. That is, D is S.S.C. if $\exists$ a monomorphism f: $D \longrightarrow W$ whenever $0 \neq W \ll_s D$[3]. A ring R is S.S.C. if R as R-Module is S.S.C. [3]. In this paper, we give purely semismall compressible Modules as a generalization of semismall compressible Modules.

An R-Module D is purely semismall compressible (P.S.S.C.) if D embedded in all its non-zero pure semismall Submodule. Equivalently, D is P.S.S.C. if $\exists$ a monomorphism $f : D \rightarrow W$ whenever $0 \neq W$ is a pure semismall Submodule of D. A ring R is P.S.S.C. if R as R-Module is P.S.S.C. Clearly, every S.S.C. Module is P.S.S.C. The converse is not true.

For example:

- Z_4 as a Z-Module is P.S.S.C. which is not S.S.C.
- An R-Module D is purely semismall simple if pure semismall Submodules of D are (0) and D.
- Every purely semismall simple Module is P.S.S.C.
- Every integral domain R is a P.S.S.C. R-Module.
- A ring R is regular ring if for every $r \in R$ $\exists$ $w \in R$ such that $r = rwr$[4].
- An R-Module D is regular Module if for every $u \in D$ and $\forall$ $r \in R$, $\exists$ $w \in R$ such that $ru = rwru$[5].
- Every regular Module D is S.S.C. iff D is P.S.S.C.
- Every Module D over regular ring R is S.S.C. iff D is P.S.S.C.

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Proposition 1. Let D_1 and D_2 be two isomorphic Modules. Then D_1 is P.S.S.C. iff D_2 is P.S.S.C.
Proof: Assume D_1 is P.S.S.C. and an isomorphism $\varphi: D_1 \to D_2$. Let $0 \neq W$ be a pure semismall Submodule of D_2. Put $K = \varphi^{-1}(W)$, thus K is a semismall submodule of D_1. Claim K is pure in D_1. Let j be an ideal of R. But f is a monomorphism gives $\varphi(jD_1 \cap K) = \varphi(jD_1) \cap \varphi(K) = j\varphi(D_1) \cap \varphi\varphi^{-1}(W) = jD_2 \cap W = jW = j\varphi(K) = \varphi(jK)$. But φ is an isomorphism, then $jD_1 \cap K = jK$. Hence K is pure in D_1. Let $f: D_1 \to K$ be a monomorphism and let $g == \varphi|_K$ thus $g: K \to D_2$ is a monomorphism and $g(K) = \varphi(\varphi^{-1}(W)) = W$, hence $g: K \to W$ is a monomorphism. Now, we have the composition $D_2 \xrightarrow{\varphi^{-1}} D_1 \xrightarrow{f} K \xrightarrow{g} W$. Let $Y = gf\varphi^{-1}$ is a monomorphism. Thus D_2 is P.S.S.C.

Proposition 2. Every non-zero pure semismall Submodule of a P.S.S.C. Module is P.S.S.C.
Proof: Assume W be a non-zero pure semismall Submodule of P.S.S.C. Module D. Assume U be a pure semismall Submodule of W. Thus, by [6, Remarks 1.2.8] U is pure in D. Then there is a monomorphism $f: W \to U$ and hence $if: U \to W$ is a monomorphism where $i: U \to W$ be inclusion homomorphism. Thus W is P.S.S.C.

Corollary 3. Every direct summand of P.S.S.C. Module is P.S.S.C.

Corollary 4. Assume D Regular Module. Every non-zero semismall Submodule of P.S.S.C. Module D is P.S.S.C.

Corollary 5. Let R be a regular ring. Every non-zero semismall Submodule of P.S.S.C. is P.S.S.C.

Remark 6. A homomorphic image (quotient) of P.S.S.C. Module is not always be P.S.S.C. For example, Z as Z-Module is P.S.S.C., but $Z/6Z \simeq Z_6$ is not a P.S.S.C. Z-Module.

Remark 7. The direct sum of P.S.S.C. Modules is not P.S.S.C.

Example 8. Let $D = Z_4 \oplus Z_2$ as a Z-Module. Clearly Z_4 and Z_2 is P.S.S.C. Z-Module. Since D is not P.S.S.C. But $D = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{1}), (\bar{1}, \bar{0}), (\bar{1}, \bar{1}), (\bar{2}, \bar{0}), (\bar{2}, \bar{1}), (\bar{3}, \bar{0}), (\bar{3}, \bar{1})\}$,$F = Z(\bar{1}, \bar{1}) = \{(\bar{0}, \bar{0}), (\bar{1}, \bar{1}), (\bar{2}, \bar{0}), (\bar{3}, \bar{1})\}$, and $G = Z(\bar{2}, \bar{1}) = \{(\bar{0}, \bar{0}), (\bar{2}, \bar{1})\}$. Clearly, $D = F \oplus G$ .Thus, F and G is a pure semismall Submodule of D, since D cannot be embedded in F (or in G). Thus D is not P.S.S.C.

An R-Module D is purely semismall prime (P.S.S.P.) if $ann(D) = ann(W)$ for each pure semismall Submodule $0 \neq W$ of D.

A Submodule W of a Module D is P.S.S.P. Submodule if $wz \in W$ with $s \in R, z \in D$ and (z) is pure semismall in D implies either $z \in W$ or $s \in [W: D]$.

Example 9. Let $D = Z_6$ as a Z-Module and $W = (\bar{2})$. W is pure semismall in Z_6 and W is P.S.S.P. Submodule of D.

Lemma 10. An R-Module D is P.S.S.P. iff (0) is a P.S.S.P. Submodule of D.
Proof: ($\Longrightarrow$) Suppose $wz = 0$ with $w \in R, z \in D$ and (z) is pure in D. Assume $z \neq 0$. Since D is P.S.S.P. then $annD = ann(z)$ and hence $w \in annD = [0: D]$ .Thus (0) is a P.S.S.P. Submodule of D.
($\Longleftarrow$) Assume (0) is a P.S.S.P. Submodule of D and $0 \neq W$ be a pure semismall Submodule of D . Let $w \in annW$. Thus $wz = 0$ for all $z \in W$, and hence $wz \in (0)$. Assume that $z \neq 0$, then $w \in [0: D] = annD$, thus $annW \subseteq annD$, so $annD = annW$, then D is P.S.S.P.

Lemma 11. Let D be a Module and every Submodule of a pure Module is pure. If W is P.S.S.P. Module, then annW is a pure semismall ideal of R for each non-zero pure semismall Submodule W of D.

Proof: Assume $0 \neq \mathrm{W}$ be a pure Submodule of D. let $\mathrm{a}, \mathrm{b} \in \mathrm{R}$ and $\mathrm{ab} \in \mathrm{annW}$. Thus $\mathrm{abW} = 0$. Suppose that $\mathrm{bW} \neq 0$. Since $\mathrm{bW} \leq \mathrm{W}$ and W is pure in D then bW is pure in D. Since D is P.S.S.P. and $\mathrm{a} \in \mathrm{annbW}$, then $\mathrm{a} \in \mathrm{annD}$, on the other hand $\mathrm{annD} = \mathrm{annW}$, so $\mathrm{a} \in \mathrm{annW}$ and hence annW is a semismall prime ideal of R. The converse of Lemma 11 is not true.

Example 12. Z_6 is not P.S.S.P. Z-Module, however $\mathrm{ann}_{\mathrm{z}}(\bar{2}) = 3\mathrm{Z}$ and $\mathrm{ann}_{\mathrm{z}}(\bar{3}) = 2\mathrm{Z}$ which are both prime ideals in Z and that $(\bar{2}), (\bar{3})$ are pure Submodule of Z_6.

Proposition 13. Every P.S.S.C. Module is P.S.S.P.
Proof: Assume D be P.S.S.C. Module and $0 \neq \mathrm{W}$ be a pure semismall Submodule of D. To show $\mathrm{annD} = \mathrm{annW}$. Let $\mathrm{x} \in \mathrm{annW}$. Thus $\mathrm{xW} = 0$. Let $\mathrm{f}: \mathrm{D} \to \mathrm{W}$ be a monomorphism, thus $\mathrm{f(xD)} = \mathrm{xf(D)} \subseteq \mathrm{xW} = 0$ implies that $\mathrm{xD} = 0$, then $\mathrm{x} \in \mathrm{annW}$ and hence $\mathrm{annD} = \mathrm{annW}$.

2. CONCLUSIONS

In this paper, we give purely semismall compressible Modules. Also, we give remarks and examples, many properties of such Modules

- D_1 is purely semismall compressible. iff D_2 is purely semismall compressible, where D_1 and D_2 be two isomorphic Modules**.**
- Every non-zero pure semismall Submodule of purely semismall compressible Module is purely semismall compressible.
- Every direct summand of purely semismall compressible Module is purely semismall compressible.
- Every non-zero semismall Submodule of regular purely semismall compressible Module is purely semismall compressible.
- Every non-zero semismall Submodule of purely semismall compressible in regular ring is purely semismall compressible.
- A homomorphic image (quotient) of purely semismall compressible Module is not always be purely semismall compressible.
- The direct sum of purely semismall compressible Modules is not purely semismall compressible.
- Every purely semismall compressible Module is purely semismall prime.

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