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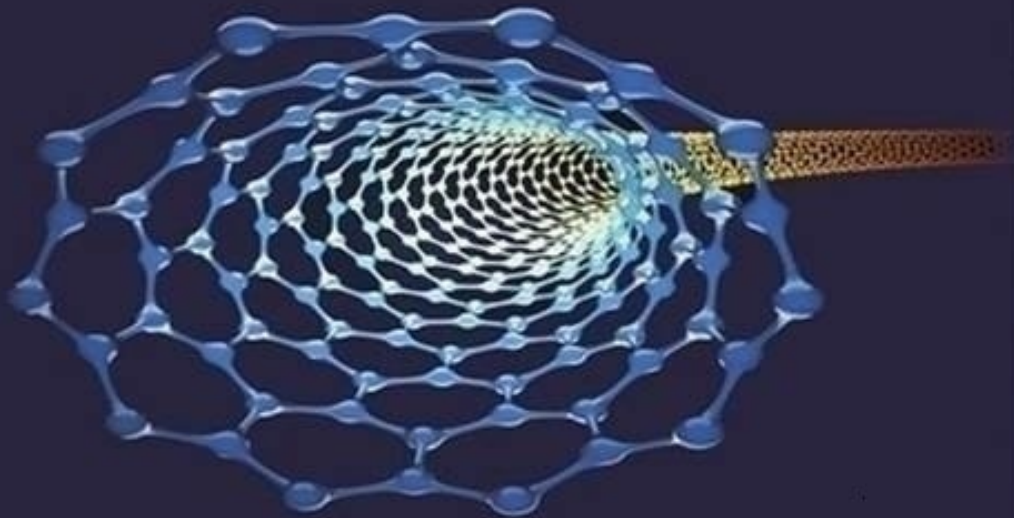

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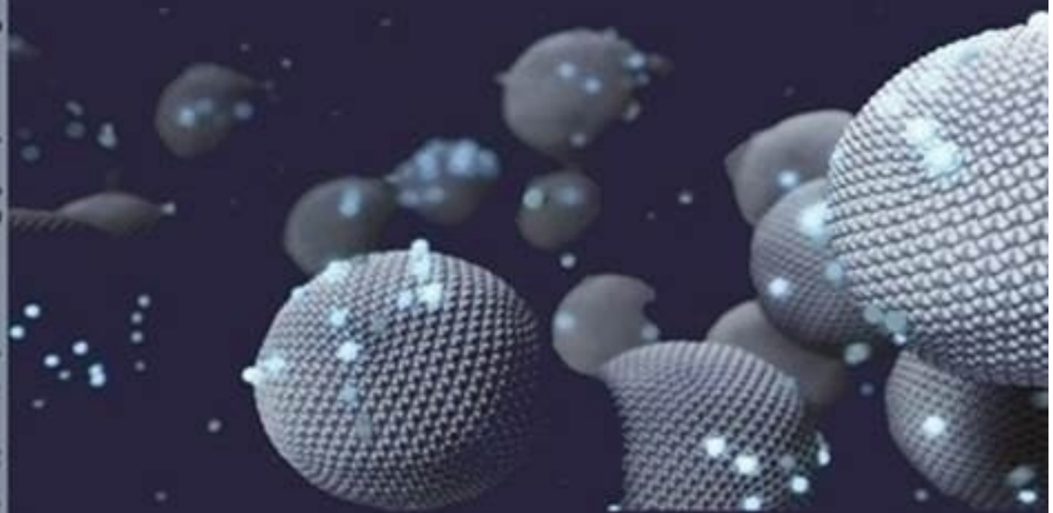

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RESEARCH PAPER

Integer-valued polynomials and binomially Noetherian rings

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A B S T R A C T:
A torsion free as a $\mathbb{Z}-$ module ring R with unit is said to be a binomial ring if it is preserved as binomial symbol

$$\binom{a}{i} := \frac{a(a-1)(a-2)\dots\big(a-(i-1)\big)}{i!},$$

for each $a \in R$ and i ≥ 0. The polynomial ring of integer-valued in rational polynomial $\mathbb{Q}[X]$ is defined by Int $(\mathbb{Z}^X) := \{h \in \mathbb{Q}[X]: h(\mathbb{Z}^X) \subset \mathbb{Z}\}$ an important example for binomial ring and is non-Noetherian ring. In this paper the algebraic structure of binomial rings has been studied by their properties of binomial ideals. The notion of binomial ideal generated by a given set has been defined. Which allows us to define new class of Noetherian ring using binomial ideals, which we named it binomially Noetherian ring. One of main result the ring Int $(\mathbb{Z}^{\{x,y\}})$ over variables x and y present as an example of that kind of class of Noetherian. In general the ring Int$(\mathbb{Z}^X)$ over the finite set of variables X and for a particular F subset in $\mathbb{Z}$ the rings Int$\big(F^{\{x_1,x_2,\dots,x_i\}}, \mathbb{Z}\big) = \{h \in \mathbb{Q}[x_1, x_2, \dots, x_i]: h(F^{\{x_1,x_2,\dots,x_i\}}) \subseteq \mathbb{Z}\}$ both are presented as examples of that kind of class of Noetherian.

KEY WORDS: Binomial ring, Integer-valued polynomial, Binomial ideal, Noetherian rings, binomially Noetherian rings.
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1. INTRODUCTION:

Binomial rings first introduced by Hall [3] in his work on nilpotent groups theory. He uses elements of a binomial ring to describe a type of generalized exponentiation of elements of any nilpotent group. There original definition is a unity torsion-free (as a Z-module) ring R is said to be binomial ring if R preserved by the binomial symbol

$$\binom{a}{i} = \frac{a(a-1)(a-2)\dots\dots\big(a-(i-1)\big)}{n} \in (R\otimes\mathbb{Q})$$

In fact, it in R for each $a \in R$ and $i \geq 0$.

Binomial rings have several useful applications in many contexts. In algebraic topology, Wilkerson [4] show that where each Adams operations over binomial rings are trivial. Then it well be a specific class of λ-rings (where λ-rings R is class of commutative with identity, equipped with operation $\lambda^i: R \to R$, for $i \geq 0$, satisfying some axioms that are satisfied by the binomial coefficients [15]) with λ-operations are presented by binomial symbol as

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$$\lambda^i(a) = \binom{a}{i},$$

for all $a \in R$ and $i \geq 0$. Also shows that on a $\mathbb{Z}$-module torsion free $\psi -$ ring R there exists a λ-ring construction that satisfy the condition (Fermat's Little Theorem)

$$\psi^P(r) \equiv r^P(\, mod\, pR).$$

Yau [14] Use above condition to see that the λ-rings are preserved under most sensible operations such as localization and completion. Also, for a binomial ring, he gives ring isomorphism between universal λ-ring $\Lambda(R)$ of R and Necklace ring $Nr(R)$ of R.

Kareem [8] as special class of λ-rings, he proves that the binomial rings are closed under localization. Applied this result to show that P −local integers $\mathbb{Z}_{(P)}$ of the ring $\mathbb{Z}$ of integers, is another example of binomial ring.

Knutson [12] built an isomorphism of ring come from a binomial ring R with a special kind of generator subset

$$R \cong \mathbb{Z}$$

Actually, that result lead to the interesting result in the topological K –theory filed. It shows the $\mathrm{K}^0(X)$ ring with a particular type of space Y is binomial ring, It gives

$$\mathrm{K}^0(Y) \cong \mathbb{Z}.$$

Similarly in representation theory if the ring $R(G)$ of representation ring is binominal ring, then G must be identity group, that is

$$R(G) \cong \mathbb{Z}.$$

In category theory Elliott in [5] As a special class of λ-rings they study the connection between λ-rings and particular type of binomial rings. He presents characterizations of binomial ring by their homomorphic image and called it quasi-binomial.

By same manner, we can introduce the new class of ideal call binomial ideal I of binomial ring R as a way we defined a class of binomial ring. A usual ideal I of binomial ring R is said to be a binomial ideal if it is preserved by binomial symbol.

$$\binom{k}{i} \in I$$

for each $k \in I$ and $i \geq 1$. Indeed there is inadequate amount of work on this kind of class of ideal (binomial ideal) to presented here . Xantcha [11] introduce a brief survey on binomial ideal.

The standard ring of polynomials of integer-valued ring with rational coefficients that send $\mathbb{Z}$ to $\mathbb{Z}$, is denote by

$$\mathrm{Int}(\mathbb{Z}^X) = \{h(X) \in \mathbb{Q}[X]: h(\mathbb{Z}^X) \subset \mathbb{Z}\}$$

where X is set of variables. Actually, it is subring of $\mathbb{Q}[X]$, where the condition that $h \in \mathbb{Q}[X]$ been integer-valued polynomial clarify as follows. First $\mathbb{Z}^X$ for each $x \in X$ set of functions given by $\underline{n}: X \to \mathbb{Z}$, where h can compute by substituting each x at any such $\underline{n}$ with the integer $\underline{n}(x)$ in h . In particular, the ring on one variable defined by

$$\mathrm{Int}\left(\mathbb{Z}^{\{x\}}\right) = \{h(X) \in \mathbb{Q}[X]: h(\mathbb{Z}) \subset \mathbb{Z}\}$$

It is called the integer-valued polynomial ring on single variable x.

In particular, the interest for the ring structure of ring $\mathrm{Int}\left(\mathbb{Z}^{\{x\}}\right)$. Polya [6] showed that the binomial polynomial

$$\binom{x}{n}$$

is basis as the Z-module of ring $\mathrm{Int}\left(\mathbb{Z}^{\{x\}}\right)$.

As several kinds of cooperations and operations the ring $\mathrm{Int}\left(\mathbb{Z}^{\{x\}}\right)$ in one variable x and their duals in complex K-theory all arising in algebraic topology. The works in that direction can be seen in [1, 2]. Algebraically this ring $\mathrm{Int}(\mathbb{Z}^X)$ over some variable X is probably one of the interesting examples of a non-Noetherian at least at our paper. (There exist an ideal of $\mathrm{Int}(\mathbb{Z}^X)$. that are not finitely generated) see [7].

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The main point of the paper therefore, to examine all deferent kinds of integer-valued polynomials rings algebraically with their binomial ideals. We establish the new class of Noetherian ring named by binomially Noetherian. These came with a certain kind of finiteness condition, namely, as a binomial all binomial ideals have to be generating finitely. As a main result $\mathrm{Int}\left(\mathbb{Z}^{\{x_1,x_2\}}\right)$ on two variables x_1 and x_2 given as examples of class of that kind of Noetherian ring. Also, by same way the Int $(F^{\{x_1,x_2\}}$, Z) for a subset $F \subseteq \mathbb{Z}$ on two variables x, x_2 is presented as another example of that class of Noetherian.

This paper is structured by the follow way. The Section 2, start by investigation about binomial rings. It shows that each $\mathrm{Int}(\mathbb{Z}^X)$ rings of integer valued polynomials on finite variables X and the Int $(F^X$, $\mathbb{Z})$ of a subset $F \subseteq \mathbb{Z}$ and over a finite of variables X. It is another example of binomial ring. Section 3 begin by establishing new notation we called ideal which is closed under binomial for a binomial ring binomial ideal, then the prove of some good properties has been given. The section 4 start with an example to explain that not all principal ideal with set of generators always can be binomial ideal. For that reason, in section 4, we define new class of ideal binomially generated by a set we named by (binomial principal ideal) alongside with some useful theory and properties. Later in this section each binomial ideal in a ring $\mathrm{Int}(\mathbb{Z}^X)$ described by polynomial ideal in ring $\mathbb{Q}[X]$. This result used to define new class of binomial rings called binomially Noetherian ring. In section 5. Then we give the ring $\mathrm{Int}\left(\mathbb{Z}^{\{x_1,x_2\}}\right)$ on variables x_1 and x_2 is an example of that class of Noetherian ring. With same identical reason we show that for subset F ⊆ Z the ring $\mathrm{Int}\left(F^{\{x_1,x_2\}},\mathbb{Z}\right)$ is another example of that class of Noetherian ring.

2. Binomial ring

Recall some notations and definitions from Ya [14]. Then we show that both rings integer-valued polynomials rings $\mathrm{Int}(\mathbb{Z}^X)$ and for particular subset F ⊆ Z the ring $\mathrm{Int}\left(F^{\{x_1,x_2,\dots,x_i\}},\mathbb{Z}\right)$ both are binomial rings. This section starts with a definition of Binomial ring.

Definition 2.1

As $\mathbb{Z}-$ module torsion-free ring R with unit is said to be binomial ring if R preserved by binomial symbol (binomial coefficient),

$$\binom{a}{i} = \frac{a(a-1)(a-2)\ldots\ldots.\left(a-(i-1)\right)}{i} \in (R \otimes \mathbb{Q}).$$

In fact, in R for each $a \in R$ and $i \geq 0$.

Example 2.2.

First, we start with some well-known examples of binomial.
The simplest one is the ring $\mathbb{Z}$ of integers.
Any characteristic 0 field R.
Each $\mathbb{Q}$-algebra.

Proposition 2.3.

For a multiplicative closed subset U. The localization ring $U^{-1}R$ of binomial R is binomial ring.

Corollary 2.4.

In ring $\mathbb{Z}$ of integers the p-local integer ring $\mathbb{Z}_{(P)}$ is a binomial ring where, p is prime.
Next began our investigation on special kinds of polynomial rings, which is called integer- valued polynomials. We will see that it is another example of class of binomial ring. The good reference for that you can see [7].

Definition 2.5.
The set of integer-valued polynomials on X that mapping $\mathbb{Z}$ to $\mathbb{Z}$, in the polynomials ring $\mathbb{Q}[X]$ on set X with rational coefficients, expressed by

$$\mathrm{Int}(\mathbb{Z}^X) = \{h(X) \in \mathbb{Q}[X]: h(\mathbb{Z}^X) \subset \mathbb{Z}\}.$$

In particular, integer-valued polynomials in one variable x id denoted by

$$\mathrm{Int}\left(\mathbb{Z}^{\{x\}}\right) = \{h(X) \in \mathbb{Q}[X]: h(\mathbb{Z}) \subset \mathbb{Z}\}$$

It is subring of polynomial ring $\mathbb{Q}[x]$.

Theorem 2.6. [14]
Let X be a set of variables. Then the ring $\mathrm{Int}(\mathbb{Z}^X)$ is an example of class of a binomial ring.
Then we can introduce another kind of polynomial ring, which is integer-valued on kind of subset on $\mathbb{Z}$.

Definition 2.7.
For a subset $F \subseteq \mathbb{Z}$, we named $h \in \mathbb{Q}[X]$ on X with $h(F^X) \in Z$ to be an integer-valued on F, with condition defined by $F^X = \mathrm{Hom}(X, F)$, as in Definition 2.4. We calculate h for any n by substitute all $x \in X$ and $k \in K$ with $n(x)$ in h. The we denoted it by

$\mathrm{Int}(F^X, \mathbb{Z}) = \{h \in \mathbb{Q}[X] : h(K^X) \subseteq \mathbb{Z}\}$, it is actually, define as a subring of $\mathbb{Q}[X]$.

In special case as sub set over the ring of integer, we have

$$\mathrm{Int}\left(F^{\{x\}}, \mathbb{Z}\right) = \{h(x) \in \mathbb{Q}[x]: h(K) \subseteq \mathbb{Z}\}.$$

Consider the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$, lead to

$$\mathrm{Int}(\mathbb{Z}^{\{x\}}) = Int(\mathbb{Z}^{\{x\}}, \mathbb{Z}).$$

Example 2.8.
For a special subset $\{0\}$, we can define,

$$\mathrm{Int}\left(\{0\}^{\{x\}}, \mathbb{Z}\right) = \{h(x) \in \mathbb{Q}[x]: h(0) \subseteq \mathbb{Z}\},$$

Specifically, $\mathrm{Int}\left(\{0\}^{\{x\}}, \mathbb{Z}\right)$ is the set of all function (polynomial) in $\mathbb{Q}[x]$ such that constant term is an integer number.

Theorem 2.9.
Consider subset F in the ring $\mathbb{Z}$ of the integers. then the ring $\mathrm{Int}\left(F^{\{x\}}, \mathbb{Z}\right)$ over x is an example of binomial ring.
Proof: First as a subring of the ring $\mathbb{Q}[x]$ the polynomial ring $\mathrm{Int}\left(F^{\{x\}}, \mathbb{Z}\right)$ is clearly $\mathbb{Z} -$ module -torsion free. To see other statement of a binomial ring, choose $g(x) \in \mathrm{Int}\left(F^{\{x\}}, \mathbb{Z}\right)$. Then

$$\binom{g}{i} = \frac{g(g-1)(g-2)\dots(g-(i-1))}{i!}$$

Select an $\underline{n} \in F^X$, then $g(\underline{n}) \in F$ implies

$$\binom{g}{i}(\underline{n}) = \frac{g(\underline{n})(g(\underline{n})-1)(g(\underline{n})-2)\dots\left(g(\underline{n})-(i-1)\right)}{i!} \in \mathbb{Q}[x], \qquad \text{for } i \geq 0$$

$$= \binom{g(\underline{n})}{i} \in \mathrm{F}.$$

Theorem 2.10.
Let F be a subset in the ring of integeres $\mathbb{Z}$. Then $\mathrm{Int}(F^X, \mathbb{Z})$ is an example of binomial ring.

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Proposition 2.11.
Any $\mathbb{Z}-$ module torsion free homomorphic image ring K onto binomial ring R is also binomial ring. .
Proof: Follow from the following

$$\varphi\binom{a}{i}=\binom{\varphi(a)}{i}=\binom{k}{i}\in K.$$

3. The binomial ideal over binomial rings:

We employ the section to investigate new class of ideal named by binomial ideal over binomial ring.

Definition 3.1.
For binomial ring R. An ideal I of R is said to be binomial ideal if

$$\binom{s}{i}\in I, \text{ for each } s\in I \text{ and } i\geq 1.$$

Example 3.2.
Consider $h(x)\in \mathrm{Int}(\mathbb{Z}^{\{x\}})$, then the set define as

$$I_{h(x)}=\{I(x)\in \mathrm{Int}(\mathbb{Z}^{\{x\}}): I(x)=h(x)g(x), \text{for some g(x)} \in \mathbb{Q}[\mathrm{x}]\}$$

is binomial ideal of $\mathrm{Int}(\mathbb{Z}^{\{x\}})$.

Example 3.3.
For $F\subseteq \mathbb{Z}$ and fixed integer $k\in F$, then the set define as

$$I_k=\{h(x)\in \mathrm{Int}(F^{\{x\}},\mathbb{Z}): h(k)=0\},$$

is binomial ideal on $\mathrm{Int}(\mathbb{Z}^{\{x\}},\mathbb{Z})$.

Next, we will see some well-known of usual ideal also satisfied by class of binomial ideal will be useful later, when we show the main result.

Proposition 3.4.
The kernel of homomorphism $\varphi: R\to E$ on binomial rings is binomial ideal.
Proof: The proof is follows from the following:

$$\varphi\left(\binom{s}{i}\right)=\binom{\varphi(s)}{i}=\binom{0}{i}=0, \qquad \text{for each } s\in R \text{ and } i\geq 1.$$

Proposition 3.5.
The inverse image of a binomial ideal by a ring homomorphism $\boldsymbol{\varphi}: \boldsymbol{R}\to \boldsymbol{E}$ is a binomial Ideal.
Proof: The proof is clear from the following

$$\varphi^{-1}\left(\binom{e}{i}\right)=\binom{\varphi^{-1}(e)}{i}=\binom{0}{i}=0,$$

for each $e\in E$ and $i\geq 1$.

Theorem 3.6. [8]
The quotient ring R/I for binomial ring R by binomial I is an example of binomial ring.

4. Binomial ideal generated by the set

We start this section with an example it is show that not all usual principal ideal is binomially principal ideal. Thus, we introduce the concept of binomial ideal generated (binomially principal ideal) by a set X.

Example 4.1.
Consider the ideal $2\mathbb{Z}$ in the ring of integers $\mathbb{Z}$, but $2\mathbb{Z}$ not binomial ideal. To see that choose an odd $n\in \mathbb{Z}$. Since $2n\in 2\mathbb{Z}$, but

$$\binom{2n}{2} = \frac{2n(2n-1)}{2} \notin 2\mathbb{Z}.$$

Commonly a usual ideal is characterized by generators set. So, for each element in the set of generators, we should examine whether it preserved by the binomial symbol. To investigating an ideal J is a binomial ideal.

Proposition 4.2.
The ideal J of ring (binomial ring) R generated by the set $Z = \{b_j\}\ j \in I$ is a binomial ideal if and only if the binomial symbol $\binom{b_j}{i} \in J$, for each $j \in I$ and $i \geq 1$.
Proof: The first part is follows from the Definition 2.1. To see the second direction, consider an element sb_j, for $s \in R$, $b_j \in Z$ and $j \in I$, then as general element we can write it,

$$y = \sum_{k=1}^{m} s_k b_{j_t}.$$

Then finite sum of products rule on $\binom{y}{i}$, we obtain

$$\binom{r_1 b_{j_1}}{p_1}\binom{r_2 b_{j_2}}{p_2}\dots\binom{r_m b_{j_m}}{p_m},$$

for $p_1 + p_2 + \cdots + p_m = i$. Thus,

$$\binom{y}{i} \in J.$$

Then it is time to establishing the notion of a binomial ideal, that is came with the set of generate.

Definition 4.3.
Let R be a binomial and S be non-empty subset of R. Then, the binomial ideal generated by the set X, we denote it by $((S))$ and define it by

$$\big((S)\big) = \cap\, \{I : S \subseteq I, I \text{ is a binomial ideal } of\ R\}.$$

So more generally, we can define the notion principal binomial ideal as

Definition 4.4.
The principal binomial ideal of R is denoted by $I = \big((a)\big)$, for some element $a \in R$.

Proposition 4.5.
Consider binomial ring R and the set

$$I = \left(\left\{\binom{s_i}{n} : n \geq 1 \text{ and } i = 1,2, \dots, k\right\}\right),$$

for $s_i \in R$. Then I is binomial ideal of R generated by the set $\{s_1, s_2, \dots, s_k\}$.

Proof: The one direction of proof is followed by Definition 4.4,
$((s_1, s_2, \dots, s_k)) \subseteq I$, for other direction by Definition 2.1,

$$\binom{s_i}{n} \in \big((s_1, s_2, \dots, s_k)\big),$$

Actually, by definition $((s_1, s_2, \dots, s_k))$ is an ideal of R. Thus, we have,

$$I \subseteq \big((s_1, s_2, \dots, s_k)\big).$$

Our first good result, which will be very useful tool, for prove the primary result. That is each binomial ideals of $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ will describe with a polynomial in $\mathbb{Q}[x]$.

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Proposition 4.6.
Consider an ideal J in $\mathbb{Q}[x]$ and define set $I = J \cap \text{Int}(\mathbb{Z}^{\{x\}})$. Thus I is binomial ideal of ring $\text{Int}(\mathbb{Z}^{\{x\}})$.
Proof: To see binomial symbol (operations) condition. Take an element $k(x) \in I$ and $i \geq 1$. Then,

$$\binom{k(x)}{n} = \frac{k(x)(k(x)-1)\ldots(k(x)-(i-1))}{i!}$$
$$= k(x).\left(\frac{(k(x)-1)\ldots(k(x)-(i-1))}{i!}\right) \in J$$

Obviously,

$$\binom{k(x)}{i} \in \text{Int}(\mathbb{Z}^{\{x\}})$$

Therefore,

$$\binom{k(x)}{n} \in I.$$

Notice that not each usual ideals of $\text{Int}(\mathbb{Z}^{\{x\}})$ can written as above description.
So, for this suppose we consider an example to see that I cannot write of the form given on Proposition 4.6.

Example 4.7.
In ring $\text{Int}(\mathbb{Z}^{\{x\}})$, choose $I = \left(\frac{x(x-1)(x-2)}{2}\right)$. Then, $\left(\frac{x(x-1)(x-2)}{2}\right).\frac{1}{3} \in J$ and $\left(\frac{x(x-1)(x-2)}{2}\right).\frac{1}{3} = \binom{x}{3} \in \text{Int}(\mathbb{Z}^{\{x\}})$, but $\left(\frac{x(x-1)(x-2)}{6}\right) \notin I$
As particular case of Theorem 3.5, we have.

Proposition 4.8.

Consider binomial ideal $I = ((d\,h(x)))$ in $\text{Int}(\mathbb{Z}^{\{x\}})$, where $d \in \mathbb{Z}$. So $h(x) \in I$ and binomially I also generated the polynomial $h(x)$, that is $I = ((h(x)))$. Actually, every binomial ideal of ring $\text{Int}(\mathbb{Z}^{\{x\}})$ can be describe in terms of polynomial ideals in $\mathbb{Q}[x]$ by form

$$I = J \cap \text{Int}(\mathbb{Z}^{\{x\}}),$$

with some ideal J in $\mathbb{Q}[x]$.

Theorem 4.9.
Each binomial ideal I in $\text{Int}(\mathbb{Z}^{\{x\}})$, can characteristic with polynomial ideal in $\mathbb{Q}[x]$ of the style form,

$$I = J \cap \text{Int}(\mathbb{Z}^{\{x\}}),$$

with some ideals define by $J = I \otimes \mathbb{Q}$ in $\mathbb{Q}[x]$.
Proof: We start by setting given ideal by $J = I \otimes \mathbb{Q}$. Then it is clear that J is an ideal of $\mathbb{Q}[x]$. Then to see the equality.

$$I = J \cap \text{Int}(\mathbb{Z}^{\{x\}}).$$

The one side of inclusion is followed $I \subseteq \left(J \cap \text{Int}(\mathbb{Z}^{\{x\}})\right)$.

To see other inclusion $J \cap \text{Int}(\mathbb{Z}^{\{x\}}) \subseteq I$. Pick $h(x) \in \left(J \cap \text{Int}(\mathbb{Z}^{\{x\}})\right)$. Thus, for some $d \in \mathbb{Z} \setminus \{0\}$ and $\bar{h}(x) \in I$, we can write $h(x) = \frac{\bar{h}(x)}{d}$. Since I is binomial ideal, by apply Proposition 3.6, we obtain $h(x) \in I$.

Theorem 4.10.

In the ring $\text{Int}(\mathbb{Z}^{X})$, every binomial ideal I can be written as $I = J \cap \text{Int}(\mathbb{Z}^{X})$, with some ideal given by $J = I \otimes \mathbb{Q}$ in polynomial ring $\mathbb{Q}[X]$.

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proof: The proof is identical to the Theorem 4.9.

Theorem 4.11.
In the ring $\mathrm{Int}(K^X, \mathbb{Z})$, every binomial ideal I can be written as $I = J \cap \mathrm{Int}(K^X, \mathbb{Z})$, with some ideal given by $J = I \otimes \mathbb{Q}$ in polynomial ring $\mathbb{Q}[X]$.

Proof: The Proof is identical to the prove of Theorem 4.9.

In next section we apply the characteristic theorem, to present the prove of primary result of our paper, which is shown that $\mathrm{Int}(\mathbb{Z}^{\{x,y\}})$ is an example of class of binomially Noetherian rings.

5. Binomially Noetherian rings

$\mathrm{Int}(\mathbb{Z}^{\{x_1,x_2\}})$ has many interesting properties and has been extensively investigated. Probably it is referenced as a nice example of non-Noetherian ring. Our main result when introduce new class of Noetherian is based on the truth that the polynomial ring $\mathbb{Q}[x_1, x_2]$ is Noetherian ring.

Definition 5.1.
A binomially Noetherian ring is a binomial ring R in which every binomial ideal I in R can be generated finitely as a binomial ideal.

Theorem 5.2.
The following conditions on binomial ring rare equivalent.

- R is binomially Noetherian: all binomial ideals of Rare finitely generated.
- If each infinite increasing sequence of binomial ideals $I_1 \subseteq I_2 \ldots . \subseteq$ in R eventually stabilizes, $I_n = I_{n+1} = I_{n+2} = \cdots$ for large n.

Proof: The proof is identical to the case in Noetherian ring, you can see [10, Theorem 11.1].

Example 5.3.
The simplest example of class of binomially Noetherian ring is the ring $\mathbb{Z}$ of integers.
Since $\binom{i}{i} = 1$, for $i \geq 1$, then for $i = 0$ and $i = 1$ in $\mathbb{Z}$, we have only 0 = ((0)) and Z = ((1)).

Example 5.4.
By same $\mathbb{Z}_{(P)}$ is also example of class of binomially Noetherian.
Same as Noetherian ring, the following results also satisfied in class of binomially Noetherian.

Proposition 5.5.
Any $\mathbb{Z}-$ module torsion free homomorphic image ring H of binomially Noetherian ring R is also binomially Noetherian ring.
Proof: Set $\phi : R \to H$ as a ring homomorphism . First by Proposition 2.10, H is binomial ring. Now, pick J in H such that $J = ((\alpha_1, \alpha_2, \ldots, \alpha_n))$, for $\alpha_i \in R$. Then $J = \varphi^{-1}(I)$ by Proposition 3.6 is a binomial ideal in R. Indeed, for all $\alpha \in J$, there exists $\beta \in I$ such that $\varphi(\beta) = \alpha$. Thus, β can be written as a linear combination of the form $\beta = s_1\beta_1 + s_2\beta_1 + \cdots + s_n\beta_n$, for some $s_i \in R$. So,

$$\alpha = \varphi(\beta) = \varphi(s_1\beta_1 + s_2\beta_1 + \cdots + s_n\beta_n) = \varphi(s_1)\beta_1 + \varphi(s_2)\varphi(\beta_2) + \cdots + \varphi(s_n)\varphi(\beta_n).$$

Therefore,

$$I = ((\,\varphi(\alpha_1), \varphi(\alpha_2), \ldots, \varphi(\alpha_n)\,)).$$

Corollary 5.6.
The quotient of a binomially Noetherian ring is also binomially Noetherian.

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Proposition 5.7.
Any localization of a binomially Noetherian ring is binomially Noetherian.
Proof: First consider homomorphism $\varphi: R \to U^{-1}R$ for multiplicative closed subset U in R. Then by apply Proposition 3.5, we claiming that,

$$\varphi^{-1}(J)(U^{-1}R) = J, \qquad \text{for a binomial ideal in } U^{-1}R.$$

One side of inclusion is followed,

$$\varphi^{-1}(J)(U^{-1}R) \subseteq J.$$

To show another side inclusion,
pick $\frac{a}{s} \in J$. So,

$$\frac{a}{s} = a\left(\frac{1}{s}\right) \in \varphi^{-1}(J)(U^{-1}R).$$

Thus, J is finitely generated.

Proposition 5.8.
Let I be a binomially Noetherian and R/I is binomially Noetherian. Then R must be also binomially Noetherian ring.
Proof: Consider an ascending sequence of binomial ideals in R as

$$J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq \cdots$$

Similarly, we obtain

$$J_1 \cap I \subseteq J_2 \cap I \subseteq \cdots \subseteq J_n \cap I \subseteq \cdots$$

and by Proposition 3.7, also we have

$$\frac{J_1}{I} \subseteq \frac{J_2}{I} \subseteq \cdots \subseteq \frac{J_n}{I} \subseteq \cdots$$

Then by hypothesis, there exist M_1, for each $n, m \geq M_1$ and M_2 such that for each $n, m \geq M_2$, we obtain

$$\frac{J_m}{I} = \frac{J_n}{I}.$$

Consequently, for $M = \max\{M_1, M_2\}$ $J_m = J_n$, for each $n, m \geq M$. by [14, Proposition p. 225],

Theorem 5.9.
The ring $\mathrm{Int}(\mathbb{Z}^{\{x_1,x_2\}})$is an example of class of binomially Noetherian.
Proof: We need see that each ideal J is finitely generated binomial ideal in $\mathrm{Int}(\mathbb{Z}^{\{x_1,x_2\}})$. Apply Theorem 3.6.11, J can be written as,

$$J = K \cap \mathrm{Int}\left(\mathbb{Z}^{\{x_1,x_2\}}\right),$$

for K be an ideal in $\mathbb{Q}[x_1, x_2]$. Consider well-known fact the polynomial $\mathbb{Q}[x_1, x_2]$ is Noetherian. So, $K = (h_1, h_2, \ldots, h_n)$ for $h_i \in \mathbb{Q}[x_1, x_2]$. That is,

$$J = (h_1, h_2, \ldots, h_n) \cap \mathrm{Int}\left(\mathbb{Z}^{\{x_1,x_2\}}\right).$$

Now pick a minimal $b_i \in N$ with

$$b_i h_i \in \mathrm{Int}\left(\mathbb{Z}^{\{x_1,x_2\}}\right),$$

and take $g_i = b_i h_i$. So $K = (g_1, g_2, \ldots, g_n)$ and $J = (g_1, g_2, \ldots, g_n) \cap \mathrm{Int}\left(\mathbb{Z}^{\{x_1,x_2\}}\right)$.
We demanding that $J = ((g_1, g_2, \ldots, g_n))$. Thus, by the Definition 3.1, we have

$$\left((g_1, g_2, \ldots, g_n)\right) \subseteq J.$$

So, to see another side of inclusion. By contradiction, we suppose

$$J \nsubseteq ((g_1, g_2, \ldots, g_n))$$

Then we have,
$((g_1, g_2, \ldots, g_n, E)) \subseteq J$ and $((g_1, g_2, \ldots, g_n, E)) \nsubseteq ((g_1, g_2, \ldots, g_n))$, for another generator E. So,

$$E = \sum_{i=0}^{n} h_i p_i = \sum_{i=0}^{n} g_i \frac{p_i}{b_i},$$

with some $p_i \in \mathbb{Q}[x_1, x_2]$, there exists $\bar{p}_i \in \mathbb{Z}[x_1, x_2]$ and $d_i \in \mathbb{Z} \setminus \{0\}$ with $p_i = \frac{\bar{p}_i}{d_i}$

So,

$$E = \sum_{i=0}^{n} g_i \frac{p_i}{b_i d_i} \in \mathrm{Int}\left(\mathbb{Z}^{\{x_1,x_2\}}\right).$$

Then we get,

$$NE = \sum_{i=0}^{n} c_i g_i \, \bar{p}_i \in \mathrm{Int}\left(\mathbb{Z}^{\{x_1,x_2\}}\right),$$

for $c_i \in \mathbb{Z}$ and N to be the least common multiple in the set$\{b_1d_1, b_2d_2, \dots, b_nd_n\}$. Thus, NE in ideal of $\mathrm{Int}(\mathbb{Z}^{\{x_1,x_2\}})$ generated with $\{g_1, g_2, \dots, g_n\}$. So, by Definition 4.1,

$$NE \in \big((g_1, g_2, \dots, g_n)\big).$$

Consequently

$$E \in \big((g_1, g_2, \dots, g_n)\big).$$

Which is contradiction and concludes that $J = ((g_1, g_2, \dots, g_n))$.
Next given as generalization of the Theorem 5.9.

Theorem 5.10.
The ring $\mathrm{Int}(\mathbb{Z}^{\{x_1,x_2,\dots,x_n\}})$ is an example of class of binomially Noetherian.
Proof: It is truth that polynomial ring $\mathbb{Q}[x_1, x_2, \dots, x_n]$ on variables $x_1, x_2, \dots, x_n$ is Noetherian. Similarly, each binomial ideal J of $\mathrm{Int}(\mathbb{Z}^{\{x_1,x_2,\dots,x_n\}})$ rewrite as

$$J = K \cap \mathrm{Int}\left(\mathbb{Z}^{\{x_1,x_2,\dots,x_n\}}\right),$$

For ideal K *in* $\mathbb{Q}[x_1, x_2, \dots, x_n]$. Thus the remain of proof is identical to the Theorem 5.9.

Corollary 5.11.
If J is principal polynomial ideal in $\mathbb{Q}[x_1, x_2, \dots, x_n]$ with set $\{h_1, h_2, \dots, h_n\}$ for some $h_i \in \mathbb{Q}[x_1, x_2, \dots, x_n]$. So principal binomial ideal $K = ((g_1, g_2, \dots, g_n))$ in $\mathrm{Int}(\mathbb{Z}^{\{x_1,x_2,\dots,x_n\}})$ can be rewrite as

$$\big((g_1, g_2, \dots, g_n)\big) = (h_1, h_2, \dots, h_n) \cap \mathrm{Int}\left(\mathbb{Z}^{\{x_1,x_2,\dots,x_n\}}\right),$$

for minimal $b_i \in \mathbb{N}$ with $b_i h_i \in \mathrm{Int}\left(\mathbb{Z}^{\{x_1,x_2,\dots,x_n\}}\right)$.

Theorem 5.12.
The ring $\mathrm{Int}(F^{\{x_1,x_2,\dots,x_n\}}, \mathbb{Z})$ on subset $F \subseteq \mathbb{Z}$ is an example of class of binomially Noetherian.
Proof: Similarly, we can rewrite each binomial ideal J in $\mathrm{Int}(F^{\{x_1,x_2,\dots,x_n\}}, \mathbb{Z})$ as

$$J = K \cap \mathrm{Int}(F^{\{x_1,x_2,\dots,x_n\}}, \mathbb{Z})$$

with some ideal $K \in \mathbb{Q}[x_1, x_2, \dots, x_n]$. So, the remain of proof is identical to the Theorem 5.9.

Proposition 5.13.
Each class of Noetherian ring is class of binomially Noetherian.
Proof: This is follows from binomial ideal is an ideal.
In general, converse of Proposition 5.13, is not right always.

Example 5.14.
The ring $\mathrm{Int}(\mathbb{Z}^{\{x_1,x_2\}})$, is by theorem 5.8, is an example of class of binomially Noetherian, but it is non-Noetherian.

Example 5.15.
By Theorem 5.8., $\mathrm{Int}(\{0\}^{\{x\}}, \mathbb{Z})$ is another example of class of binomially Noetherian. But Strickland in [9, Example 18.3], saw that the ideal $K = xQ[x]$ is not principal in $\mathrm{Int}(\{0\}^{\{x\}}, \mathbb{Z})$.

6. CONCLUSIONS

In this paper, we introduce Binomially Noetherian, a new class of Noetherian with the property of their binomial ideal. We describe each binomial ideal of $\mathrm{Int}(\mathbb{Z}^X)$, in a team of polynomial ideals of $\mathbb{Q}[X]$.. Use this result to Show that the integer valued polynomial ring $\mathrm{Int}(\mathbb{Z}^{\{x_1,x_2\}})$ on sets of variables X and the integer valued polynomial ring over subset $\mathrm{Int}(F^{\{x\}}, \mathbb{Z})$ for $F \subseteq \mathbb{Z}$ are both examples of Noetherian rings.

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