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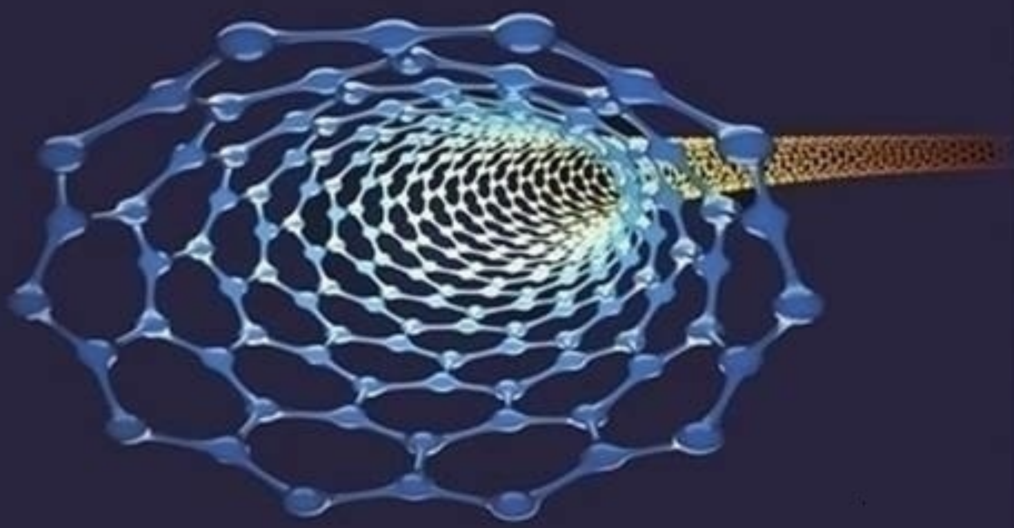

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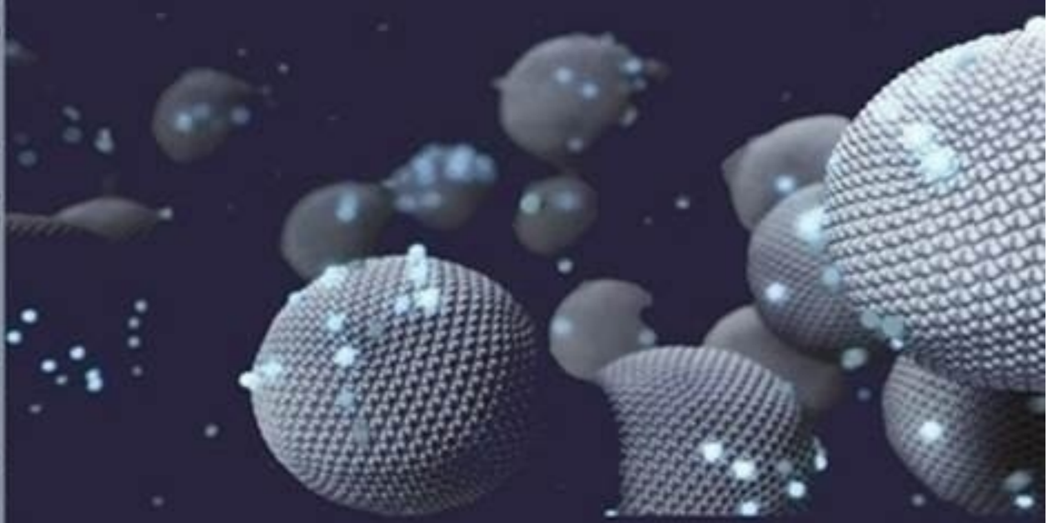

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RESEARCH PAPER

New Results in Bi- Domination in Graphs

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A B S T R A C T:
In this paper, some new results are introduced for the bi-domination in graphs. Some properties of bi-domination number and bounds according to maximum, minimum degrees, order, and size have been determined. The effects of removing a vertex and removing or adding an edge are discussed on the bi-domination number of a graph. This study is important to know affected graphs by the deletion or addition of components.

KEY WORDS: domination number, bi-domination number, minimum dominating set.
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1. INTRODUCTION:

Consider $G=(V,E)$ be a graph without an isolated vertex where V is the vertex set of order n and E is the edge set of size m. the degree of a vertex of any graph G is the number of edges incident on this vertex. It is denoted by $deg(v)$, where $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degrees of vertices in a graph G respectively. The *open* neighborhood of a vertex v is the set $N(v)=\{u\in V,\ uv\in E\}$ while the closed neighborhood is $N[v]=N(v)\cup\{v\}$. Consider a vertex $v\in\ V$. A set $D\ \subseteq V(G)$ is called a dominating set (DS) in the graph $G\ if\ N\ (v)\cap D\neq\emptyset, \forall v\in V-D$[17]. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality over all DS in G. Domination deals with various fields in graph theory as a topological graph [5], fuzzy graph [18], labelled graph [15,20]. Also, there is a study of domination polynomial of certain graph as in [8]. Conditions are imposed on the dominant number in order to fit the problems for which it is intended to solve. Some of the conditions mentioned earlier are put on the dominating set as in [6,16]. Or by putting conditions out of the DS as in [4,14]. And some definitions included both methods like [7,9]. Al-Harere and Breesam [1] are introduced a new model of domination under the condition that every vertex dominates exactly two vertices called bi-domination. These vertices do not belong to the DS. In [2] five new definitions of domination have been presented, which they are modified versions of bi-domination: “connected bi-domination”, “total bi-domination”, “connected independent bi-domination”, “restrained domination”, and “complementary tree bi-domination”. The lower and upper bounds are calculated for the size of graphs having these parameters. In [3] bi- domination in spinner graph is determined. In this paper, some properties for bi- DS set are introduced.
Also, the size of a graph which has bi-domination number is determined. We proved that every bi- DS, is a minimal bi- DS. Moreover, several bounds are founded on this number for a graph G, according to its order, set of pendants, minimum and maximum degrees of its vertices.

Finally, changes that may occur in bi-domination number were discussed when a vertex or an edge is added to a graph, or when it is deleted. For graph theoretic terminology we refer to [11]. An excellent treatment of several topics in domination can be found in [10,12,13].

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2. Bi-dominating sets

Definition 2.1. [1] Consider $G = (V, E)$ be a finite, simple, and undirected graph without isolated vertex. A subset $D \subset V(G)$ is a $bi - DS$, if $\forall\, v \in D$ dominates exactly two vertices in the set $V - D$, such that $|N(v) \cap \{V - D\}| = 2$. The minimum bi-dominating set of G is denoted by $\gamma_{bi}(G)$-set. The minimum cardinality of all $bi - DS$ is called bi-domination number and denoted by $\gamma_{bi}(G)$.

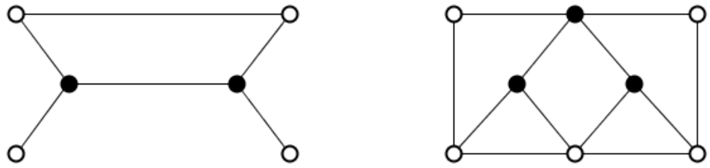

Figure 1: $bi - DS$ in graphs

Observation 2.2. [1] Let G be a finite simple undirected graph of order n with a $bi - DS$ D and γbi(G). We have

- The order of a graph G is $n; n \geq 3$.
- $\delta(G) \geq 1$, $\Delta(G) \geq 2$.
- Every $v \in D$, $deg(v) \geq 2$.
- Every support vertex $v, v \in D$.
- $\gamma(G) \leq \gamma_{bi}(G)$.

Observation 2.3. Assume that a graph G has $bi - DS$, then $\gamma_{bi}(G) \leq n - p$ where p is the number of pendant vertices in G.

Observation 2.4. If there is a component K_2 in a graph G, then G has no $bi - DS$.

Observation 2.5. If G has support vertices adjacent to more than 2 pendants vertices in a graph G, then G has no $bi - DS$.

Proposition 2.6. Let G be any graph which has a $bi - DS$, then $\gamma_{bi}(G) = 1$ if and only if G is an either $P_3 or\ K_3$.

Proof: If $\gamma_{bi}(G) = 1$, then G has a $bi - DS$ contains one vertex, this vertex dominates two vertices in $V - D$. Thus, G is connected and has only three vertices therefore, G is an either $P_3 or\ K_3$. Conversely, it is clear.

Theorem 2.7. Assume that G be a graph having $\gamma_{bi}(G)$ then the size m of G is

$$2\gamma_{bi} \leq m \leq \frac{n^2-n}{2} + {\gamma_{bi}}^2 - n\gamma_{bi} + 2\gamma_{bi}$$

Proof: Consider D to be a γ_{bi} - set of G, so we prove the required by discussing two different cases as follows.

Case 1. Firstly, to prove $2\gamma_{bi} \leq m$, let $G[\,D\,]$ and $G[\,V - D]$ be two null graphs. Hence, G contains as few as possible edges. Now, by the definition of $bi - DS$ there exist exactly two edges incident to all vertices in D. Thus, the number of edges is $2\ |D| = 2\gamma_{bi}$. Therefore, $m \geq 2\gamma_{bi}$.

Case 2. To prove the upper bound, this case occurs where the two induced subgraphs of the sets D and $V - D$ are complete, so let m_1 *and* m_2 be the number of edges of two induced subgraphs *of the sets* D *and* $V - D$ respectively. Thus,

$$m_1 = \frac{|D|\,|D-1|}{2} = \frac{\gamma_{bi}(\gamma_{bi}-1)}{2} \quad and\ m_2 = \frac{|V-D||V-D-1|}{2} = \frac{(n-\gamma_{bi})(n-\gamma_{bi}-1)}{2} .$$

And according to case 1 we have

$$m_3 = 2|D| = 2\gamma_{bi}$$

So, in this case

$$m = m_1 + m_2 + m_3$$

Hence,

$$m = 2|D| + \frac{|D|\,|D-1|}{2} + \frac{|V-D||V-D-1|}{2} = 2\gamma_{bi} + \frac{\gamma_{bi}(\gamma_{bi}-1)}{2} + \frac{(n-\gamma_{bi})(n-\gamma_{bi}-1)}{2}.$$

In general

$$m \le m_1 + m_2 + m_3$$

Theorem 2.8. Assume that a graph G has γ_{bi}, then

$$\lceil n/3 \rceil \le \gamma_{bi}(G) \le n-2$$

Proof. Let D be a γ_{bi}- set of G, and let the vertices $v_i, v_j \in D$. Then there are two different cases as the following.

Case1. If $N(v_i) \cap N(v_j) \cap (V-D) = \emptyset$, so each vertex in the set $V-D$ is dominated by exactly one vertex in the set D, and $\gamma_{bi}(G) = n/3$.

Case2. If $N(v_i) \cap N(v_j) \cap (V-D) \neq \emptyset$, this means exists one vertex or more being dominated by the same vertex in D which means $\gamma_{bi}(G) > \lceil n/3 \rceil$.

Thus, $\gamma_{bi}(G) \ge \lceil n/3 \rceil$. The upper bound is obvious.

Corollary 2.9. Consider G be a graph having a γ_{bi}, then

1. $\gamma_{bi}(G) \ge \left\lceil \frac{n}{\delta+2} \right\rceil, \delta \ge 1$.
2. $\gamma_{bi}(G) \ge \left\lceil \frac{n}{\Delta+1} \right\rceil, \Delta \ge 2$.

Theorem 2.10. Every $bi-DS$ is a minimal $bi-DS$.

Proof. Assume that the set D be any $bi-DS$ in a graph G. Assume that the set D is not a minimal $bi-DS$, so there is at least one vertex say $v \in D$ such that $D-\{v\}$ is a $bi-DS$. Now we discuss the deletion cases as follows.

Case 1. Assume that there are two vertices that are dominated by the vertex v is not dominated by the other vertex. Then the set $D-\{v\}$ is not a $bi-DS$ and this is a contradiction.

Case 2. If there are one or more vertices in $D-\{v\}$ which dominate the two vertices in $V-D$ that are adjacent to the vertex v, then we discuss which vertices are dominating vertex v. Now, if the set $D-\{v\}$ has no a vertex dominating the vertex v, then $D-\{v\}$ is not a $bi-DS$, so this is a contradiction too. Otherwise, the vertex v is dominated by at least one vertex say w in the set $D-\{v\}$. Therefore, w dominates at least three vertices in the set $V-(D-\{v\})$. Thus, the set $D-\{v\}$ is not a $bi-DS$ and this is a contradiction. From all cases above, the set $D-\{v\}$ is not a $bi-DS$, so, D is the minimal $bi-DS$.

3. Changing and unchanging of bi-domination number:

Throughout this section the effects on $\gamma_{bi}(G)$ when G is modified by deleting a vertex or deleting or adding an edge are discussed.

If $G-v$ has a $bi-DS$, then the three partitions of the vertices of G are:

$$V^0 = \{ v \in V : \gamma_{bi}(G-v) = \gamma_{bi}(G)\}.$$
$$V^+ = \{ v \in V : \gamma_{bi}(G-v) > \gamma_{bi}(G)\}.$$
$$V^- = \{ v \in V : \gamma_{bi}(G-v) < \gamma_{bi}(G)\}.$$

In the same manner the edge set can be classification into

$$E_*^0 = \{e \in E : \gamma_{bi}(G*e) = \gamma_{bi}(G)\}$$
$$E_*^+ = \{e \in E : \gamma_{bi}(G*e) > \gamma_{bi}(G)\}$$
$$E_*^- = \{e \in E : \gamma_{bi}(G*e) < \gamma_{bi}(G)\}, \text{ where } * = \begin{cases} - & , \quad e \in G \\ + & , \quad e \in \bar{G} \end{cases}$$

Theorem 3.1. For graph G having a unique γ_{bi}- set, there exists a vertex v such that if $G-v$ has a bi-dominating set then v belongs to $(V^0 \cup V^+ \cup V^-)$

Proof. By assumption there is a unique γ_{bi}- set say D then there are two different cases as follows.

(a) When $v \in D$, vertex v dominates two vertices say w_1 and w_2 in $V-D$ then there are three cases

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Case1. $v \in V^0$, this case occurs in two cases as the following.
i. If vertex v dominates w_1 and w_2 such that only one vertex say $w_1 \in pn[v, D]$ and it is adjacent to exactly two vertices in $V - D$, then we can add this vertex (w_1) to set $D - \{v\}$. It is obvious that $\{D - v \cup \{w_1\}\}$ is a $bi - DS$ and $\gamma_{bi}(G - v) = \gamma_{bi}(G)$. (For example, see Fig. 2).

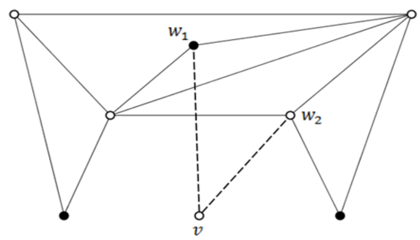

Figure 2: $bi - DS$ of $G - v$

ii. If the both vertices w_1 *and* $w_2 \in pn[v, D]$, such that w_1 is adjacent to w_2 and to other vertex in $V - D$, then we can add vertex w_1 to set $D - \{v\}$. It is obvious that $D - \{v\} \cup \{w_1\}$ is a minimal $bi - DS$ and $\gamma_{bi}(G - v) = \gamma_{bi}(G)$. (For example, see Fig. 3).

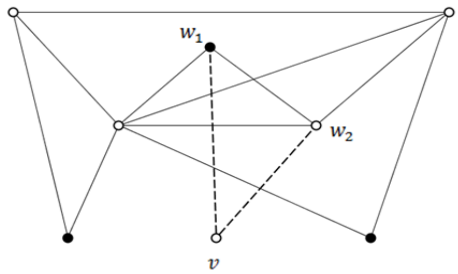

Figure 3: $bi - DS$ of $G - v$

Case 2. A vertex $v \in V^+$ if both vertices w_1 *and* $w_2 \in pn[v, D]$. So that the two vertices w_1 and w_2 are not dominated by vertices of $D - \{v\}$, and each one of w_1 and w_2 is exactly adjacent to two vertices in $V - (D - \{v\})$ regardless the possibility that they are adjacent to each other or not. Then we can add the vertices w_1 and w_2 to the set $D - \{v\}$ and it is obvious that $\{D - \{v\} \cup \{w_1, w_2\}\}$ is a minimal $bi - DS$ and $\gamma_{bi}(G - v) > \gamma_{bi}(G)$. (For example, see Fig. 4).

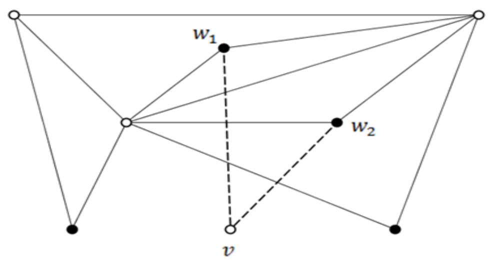

Figure 4: $bi - DS$ of $G - v$

Case 3. A vertex $v \in V^-$ if the two vertices w_1 *and* w_2 are dominated by other vertices in D, then $\gamma_{bi}(G - v) < \gamma_{bi}(G)$.
(b) When $v \in V - D$, then there are two cases
Case1. $v \in V^0$
The vertex in D that dominates the vertex v say u dominates another vertex in $V - D$ say w, if w is adjacent to exactly one vertex in $V - D$, then we can take the vertex w instead of the vertex u.
Case 2. $v \in V^-$, if the vertex v is dominated by more one vertex in D say the set $M = \{v_1, v_2, .. v_i, \dots, v_k\}$, and there is a vertex in M say v_i such that all other vertices in M are adjacent to it. G is $bi - DS$ and $v_i \in D$,

thus there is a vertex $v \neq u \in V - D$ which dominates by the vertex v_i and suppose that this vertex is already dominated by another vertex in D, then the vertex v_i will take place the vertex v in the set $V - D$. Therefore, $\gamma_{bi}(G - v) < \gamma_{bi}(G)$. (For example, see Fig. 5).

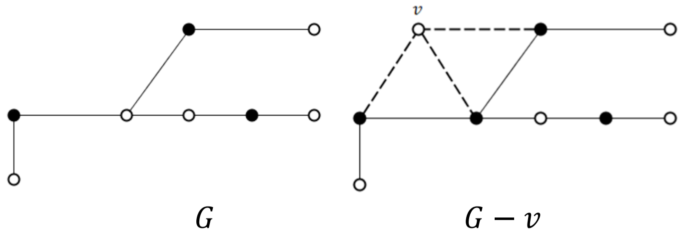

Figure 5: $bi - DS$ in G and $G - v$

Theorem 3.2. Assume that $G = (V, E)$ be a graph with a minimum $bi - DS$, and $e \in \bar{G}$, if $G + e$ has $bi - DS$ then either $e \in E_+^0$ or $e \in E_+^+$.

Proof. Let D be a minimum $bi - DS$. If e is added to $G[D]$or to $G[V - D]$, it is obvious that the set D is not influenced by this addition, which means $\gamma_{bi}(G) = \gamma_{bi}(G + e)$. Thus, $e \in E^0$.
If e is added to G, such that one vertex incident with e say v belongs to D and the other vertex say w belongs to $V - D$ and let u and z are two vertices in $V - D$ which are dominated by to vertex v. When $G + e$ has a $bi - DS$ then there are two cases as the following.
Case 1. $e \in E_+^0$: There are three different cases as the following.
i. If u and $z \in pn[v, D]$ and there is one or more induced subgraph K_3 in G then there are three different cases as the following.
a) If v, u, and z are the vertices of K_3, in $G + e$, where $e = vw$, we can take either u or z instead of the vertex v in D if this vertex of degree two in G (as shown in Fig. 6).
b) If the two vertices u and z are not adjacent and w is a vertex of the induced subgraph K_3 then in $G + e$ we can add w instead of its dominating vertex in D if both of them are of degree two in G.
(as shown in Fig. 7)
c) If there are two induced subgraphs K_3, then we can combine the two cases above.

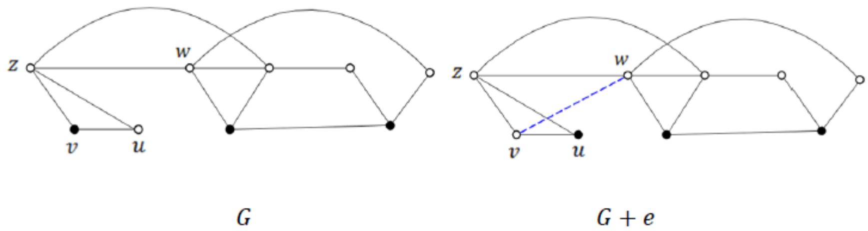

Figure 6: A minimum $bi - DS$ in G and $G + e$

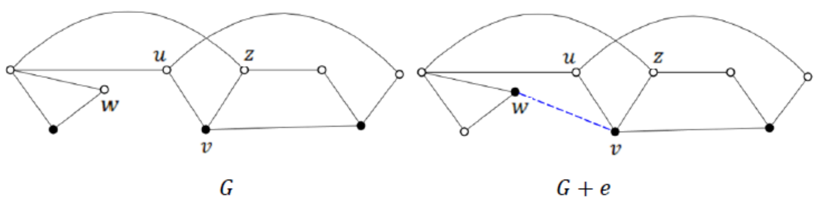

Figure 7: A minimum $bi - DS$ in G and $G + e$
ii. If one of the vertices say z is adjacent to some vertices in D say h, then there are two different cases:
a) if $deg\ (u) = deg\ (v) = 2$ in G, then we can take u instead of v in D. (as shown in following Fig. 8).

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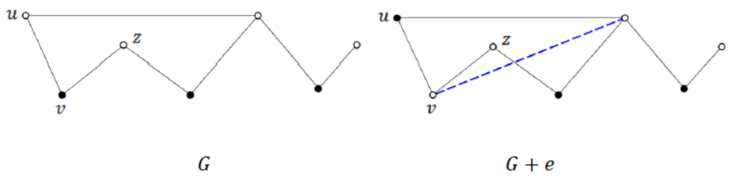

Figure 8: A minimum $bi - DS$ in G and $G + e$

b) If $deg\ (u) = deg\ (w) = 3$ in $G + e$ such that u and w are adjacent then we can take $u\ and\ w$ instead of $v\ and\ h$ respectively if $deg\ (v) = deg\ (h) = 2$ in G. (see Fig. 9).

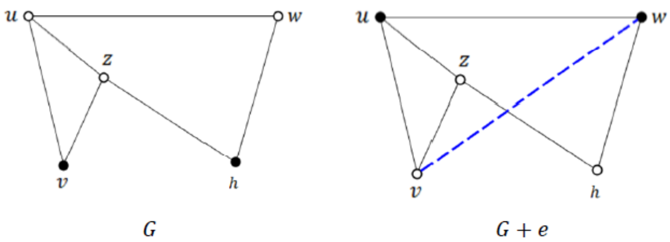

Figure 9: A minimum $bi - DS$ in G and $G + e$

iii. If u and z are dominated by other vertices of D, then four cases are discussed as follows.
a) Add w to set D instead of its dominating vertex, if w is a vertex as in case 1(i)(b)

b) Having a component of G such that v, u, z and, h, are its vertices, where both $v\ and\ h$ dominate u and z. So, we take u and z instead of v and h as the dominating vertices. (as illustrated in Fig. 10).

c) If h and c dominate u and z respectively in addition to v. Also, v and h are isolated vertices in $G[D]$, assume $w \in pn[D, h]$, $deg(w) = 3$ in $G + e$. So, w will replace h in D. (as illustrated in Fig. 11).

d) Assume h dominates both u and x and c dominates z and another vertex, such that $v\ and\ h$ are isolated vertices in $G[D]$and u is an isolated in $G[V - D]$, so $u\ and\ x$ will replace $v\ and\ h$ respectively in D. (as illustrated for example in Fig. 12).
In all sub cases $a, b, c,$and d $\gamma_{bi}(G) = \gamma_{bi}(G + e)$
So, in case 1 $\gamma_{bi}(G) = \gamma_{bi}(G + e)$.

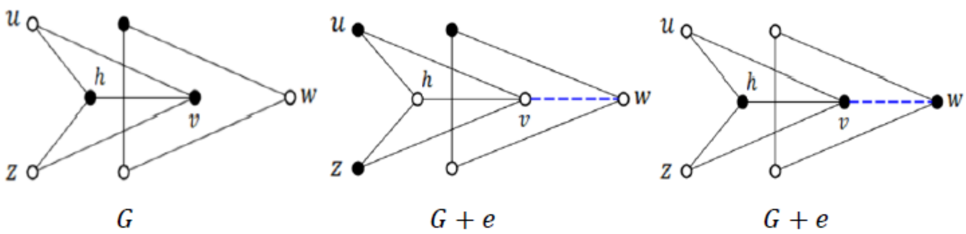

Figure 10: A minimum $bi - DS$ in G and $G + e$

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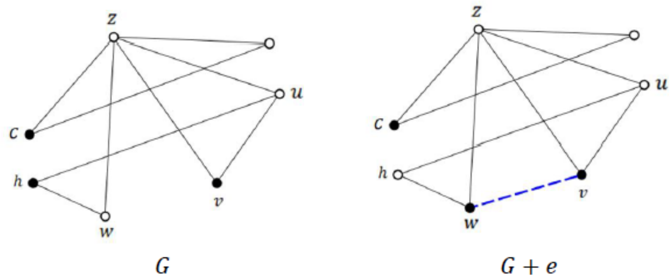

Figure 11: A minimum $bi - DS$ in G and $G + e$

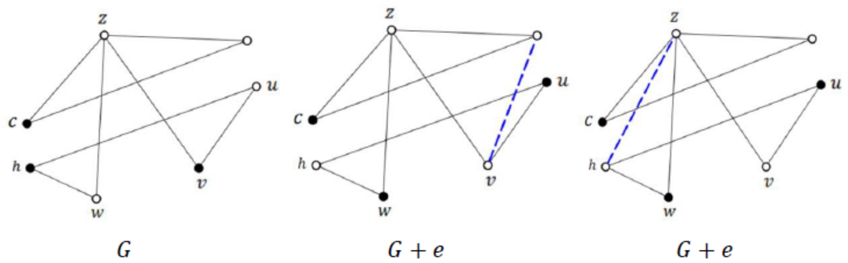

Figure 12: A minimum $bi - DS$ in G and $G + e$

Case 2. $e \in E_{+}^{+}$: This case occurs in three different cases below.
i) If both u and $z \in pn[v, D]$ in G , we can add u and z to D instead of the vertex v if u and z are not adjacent to each other and v is isolated in $[D]$. Also, $deg\ (u) = deg\ (z) = 1$ in G$[V - D]$. (as an example, see Fig. 13).

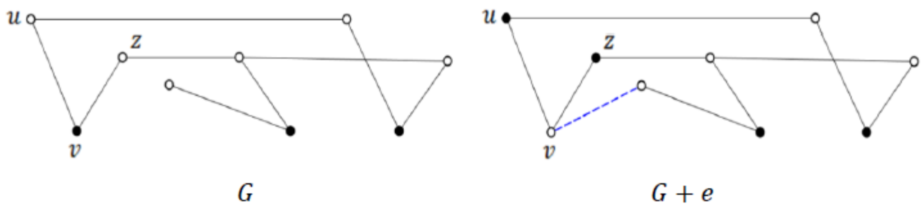

Figure 13: A minimum $bi - DS$ in G and $G + e$

ii) If one of two vertices say u satisfies that $|N(u) \cap (V - D)| = 2$, then in $G + e$, we add vertex u to the set D, so we get the result. (for example, see Fig.14).
Hence, in case 2 $\gamma_{bi}(G + e) > \gamma_{bi}(G)$

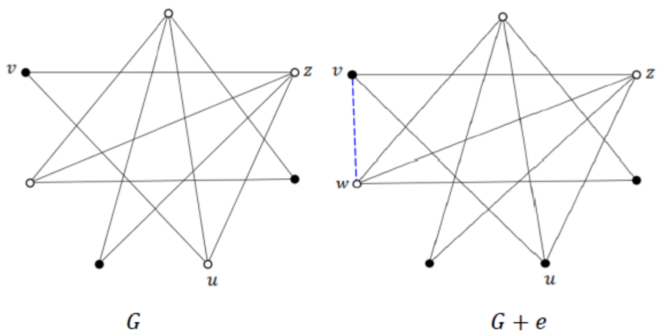

Figure 14: A minimum $bi - DS$ in G and $G + e$

Theorem 3.3. Let $G = (V, E)$ be a graph with a minimum $bi - DS$, and $e \in E(G)$, if $G - e$ has $bi - DS$ then, $E_-^* \neq \emptyset, where *= 0\ or\ -, or\ +.$

Proof. Let D be a $minimum\ bi - DS$ .There are three cases to show the set E_-^* is not an empty set as follows.
Case 1. $E_-^0 \neq \emptyset$.
If e is deleted from $G[D]$or from $G[V - D]$, it is obvious that the minimum $bi - DS$ is not influenced for this change, which means $\gamma_{bi}(G - e) = \gamma_{bi}(G)$. Thus, $e \in E_-^0$. Hence, $E_-^0 \neq \emptyset$.
Case 2. $E_-^+ \neq \emptyset$
Let $e = vu$, where $v \in D\ and\ u \in V - D$ and let $v \neq w \in V - D$ that it is dominated by the vertex v and suppose that u and w are adjacent. Also, u is adjacent to exactly two other vertices in $V - D$ and w is adjacent to only one vertex in $V - D$, then $(D - \{v\}) \cup \{u, w\}$ is a minimum $bi - DS$. Thus, we get the result. (see Fig. 15).

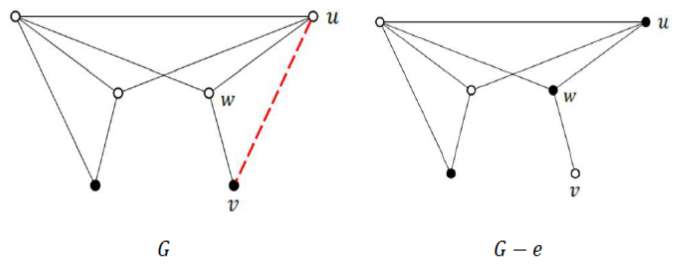

Figure 15: A minimum $bi - DS$ in G and $G - e$

Case 3. $E_-^- \neq \emptyset$
Let $e = vu$, where $v \in D\ and\ u \in V - D$. In $G - e$, if there is a vertex in D say w such that w is adjacent to only v in set D. So, $(w) \cap D - \{v\} = \emptyset$, and there is a vertex or more in $D - \{w\}$ dominating the two vertices which are dominated by w, then $D - \{w\}$ is a $bi - DS$. Therefore, $E_-^- \neq \emptyset$ (as an example, see Fig.16).

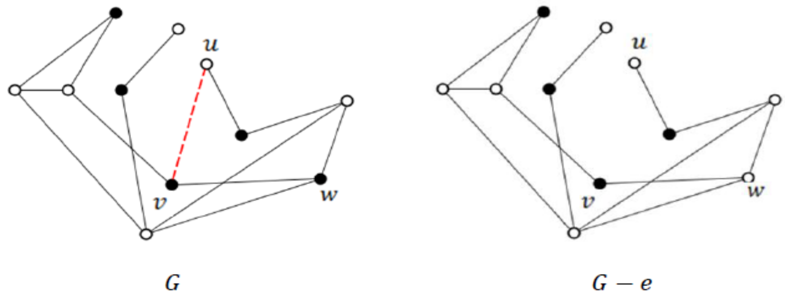

Figure 16: A minimum $bi - DS$ in G and $G - e$

4. CONCLUSIONS

The domination type called bi-domination is one of the domination types can be calculated in a connected and disconnected graph. This definition can be determined if applicable or not depending on graph size. Every bi-dominating set is a minimal bi-dominating set. Also, bi-domination number could be maintained when a vertex is deleted, or when we add or delete an edge from the graph.

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