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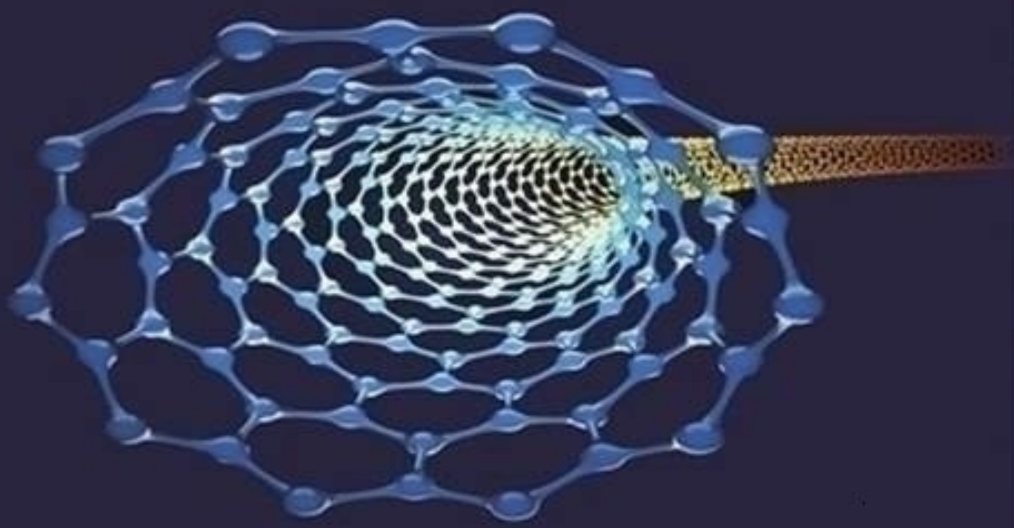

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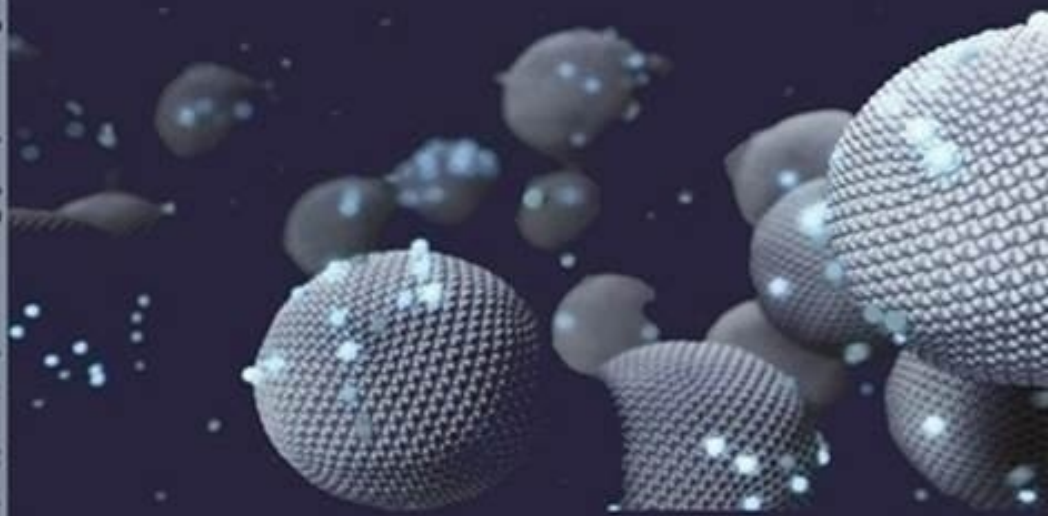

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RESEARCH PAPER

Hopf Bifurcation Analysis of a Chaotic System

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A B S T R A C T:
This paper is devoted to studying the stability of the unique equilibrium point and the occurrence of the Hopf bifurcation as well as limit cycles of a three-dimensional chaotic system. We characterize the parameters for which a Hopf equilibrium point takes place at the equilibrium point. In addition, the system has only one equilibrium point which is $E_0 = (0,0,0)$. It was proved that E_0 is asymptotically stable and unstable when $\alpha < \frac{-13}{7}$ and $\alpha > \frac{-13}{7}$, respectively. Moreover, for studying the cyclicity of the system, two techniques are used which are dynamics on the center manifold and Liapunov quantities. It was shown that at most two limit cycles can be bifurcated from the origin. All the results presented in this paper have been verified by a program via Maple software.

KEY WORDS: Chaotic System, Hopf Bifurcation Analysis, equilibrium point.
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1. INTRODUCTION:

We consider the following system of differential equations

$$\frac{dU}{dt} = AU + F(U,\mu) \tag{1}$$

where variable $U \in \mathbb{R}^3$, F is an analytic function, parameter $\mu \in \mathbb{R}^k$ and A is the square matrix has two complex eigenvalues $\alpha \pm i\beta$ $(\beta \neq 0)$ and a non-zero eigenvalue γ, such that $F(0;\mu) = D_U(0;\mu) = 0 \quad \forall\mu$, where $D_U(0;\mu)$ is the determinant of Jacobian matrix of $F(U;\mu)$ $at\ U = 0$.

A sufficient condition of Hopf bifurcation for the three-dimensional system (1) (having a nonzero with two pure imaginary eigenvalues) is illustrated below:

Let

$$\lambda^3 - T\,\lambda^2 - K\,\lambda - D = 0 \tag{2}$$

be the characteristic polynomial of the linearized system (1) at the origin, where

$$\begin{aligned} T &= \textstyle\sum_{i=1}^{3} a_{i,i} \quad \text{(Trace of the Jacobian matrix of system (1) at the origin),} \\ D &= \text{Determinant of the Jacobian matrix of system (1) at the origin,} \\ K &= -(A_1 + A_2 + A_3); \end{aligned}$$

where $A_i = a_{j,j}a_{k,k} - a_{j,k}a_{k,j}$, $i, j, k = 1,2,3$, $i \neq j \neq k$ are elements of the Jacobian matrix of system (1) at the origin [1], [2] and [3]. Then, the Hopf bifurcation takes place at a point (which is called a Hopf point) where

$$TK + D = 0 \quad (T \neq 0\ \&\ K < 0) \tag{3}$$

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Moreover, suppose that system (1) has a critical point (u_0, μ_0), then this system has a Hopf bifurcation if $D_u f(u_0, \mu_0)$ has a simple pair of pure imaginary eigenvalues with no other eigenvalues with zero real parts and $\frac{d}{d\mu} Re(\lambda_{2,3}(\mu))|_{\mu=\mu_0} = d \neq 0$ are satisfied, where $Re(\lambda_{2,3}(\mu))$ denotes the real part of the complex eigenvalues which is smooth function of μ. Then, there is a unique three-dimensional center manifold passing through (u_0, μ_0) in $\mathbb{R}^3 \times \mathbb{R}$ and a smooth system of coordinates (preserving the planes $\mu = const.$) for which the Taylor expansion of degree three on the center manifold is given by

$$\begin{pmatrix} \dot{u} \\ \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \left(d\,\mu + a\,(u^2+v^2)\right) u - \left(\omega + c\,\mu + b\,(u^2+v^2)\right) v \\ \\ \left(\omega + c\,\mu + b\,(u^2+v^2)\right) u + \left(d\,\mu + a\,(u^2+v^2)\right) v \end{pmatrix} \tag{4}$$

If $a \neq 0$, there is a surface of periodic solution in the center manifold which has quadratic tangency with the eigenspace of $\lambda\ (\mu_0)$, $\overline{\lambda\ (\mu_0)}$ agreeing to second order with the parabolic $\mu = -\frac{a}{d}\ (u^2+v^2)$. If $a < 0$, then these periodic solutions are stable limit cycles, the bifurcation is of type Supercritical Hopf bifurcation. While if $a > 0$, the periodic solutions are repelling (unstable limit cycles), the bifurcation is of type Subcritical Hopf bifurcation [4] and [5].

By the time rescaling $\tau = \beta t$ and a linear change of coordinates, system (1) can be written in the form

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \alpha_1 u - v \\ \alpha_1 v + u \\ \lambda\, w \end{pmatrix} + \begin{pmatrix} F_1(u,v,w;\mu) \\ F_2(u,v,w;\mu) \\ F_3(u,v,w;\mu) \end{pmatrix} \tag{5}$$

where $\alpha_1 = \frac{\alpha}{\beta}$, $\lambda = \frac{\gamma}{\beta}$, $F_i(u,v,w;\mu) = \sum_{k=2}^{\infty} F_i^{\,k}(u,v,w;\mu)$, $i = 1,2,3$ *and* $F_i^{\,k}(u,v,w;\mu)$ are polynomials that are homogeneous of degree k. The Hopf point at the origin of equation (5) has two pure imaginary eigenvalues, $\pm i$, and a nonzero eigenvalue λ when $\alpha_1 = 0$. A good source of Hopf bifurcation in $\mathbb{R}^n$ is [6]. At $\alpha_1 = 0$, system (5) can be written of the following form

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -v \\ u \\ \lambda\, w \end{pmatrix} + \begin{pmatrix} F_1(u,v,w;\mu) \\ F_2(u,v,w;\mu) \\ F_3(u,v,w;\mu) \end{pmatrix} \tag{6}$$

where F_1, F_2 and F_3 are real analytic functions on the neighborhood of the origin in $\mathbb{R}^3$ and with their derivatives vanish at the origin. Since system (6) has two eigenvalues with zero real part $\alpha_1 = 0$, then system (6) has a local 2-dimensional centre manifold, $W^c(0)$ [7]. This manifold is invariant and there exists a function h of class C^K, $k \geq 1$ in a small neighbourhood of the origin such that $h(0,0;\ \mu) = D\,h(0,0;\ \mu) = 0$, where $D\,h(0,0;\ \mu)$ is a Jacobian matrix of h at the origin. The 2-dimensional centre manifold, $W^c(0)$, is defined by

$$W^c(0) = \{(u, v, h(u,v;\ \mu);\ \mu) \in \mathbb{R}^3 : (u,v) \in \text{a small neighborhood of the origin}\}$$

In the third component of equation (6), after inserting $w = h(u,v;\ \mu)$ and using the chain rule, the following equation is obtained, which is useful to find the function h.

$$D\,h(0,0;\ \mu)\begin{pmatrix}-v+F_1(u,v,h(u,v;\ \mu);\mu)\\ u+F_2(u,v,h(u,v;\ \mu);\mu)\end{pmatrix}=\lambda\,h(u,v;\ \mu)+F_3(u,v,h(u,v;\ \mu);\mu) \tag{7}$$

In the first two components of equation (6), after substituting $w=h(u,v;\ \mu)$, the following reduced system to the center manifold is obtained; its linear part is of center-focus type.

$$\begin{pmatrix}\dot{u}\\ \\ \dot{v}\end{pmatrix}=\begin{pmatrix}-v\\ \\ u\end{pmatrix}+\begin{pmatrix}F_1(u,v,h(u,v;\ \mu);\mu)\\ \\ F_2(u,v,h(u,v;\ \mu);\mu)\end{pmatrix} \tag{8}$$

To find the Liapunov quantities of system (8), we seek a Lyapunov function of the form

$$V(u,v)=u^2+v^2+\sum_{k=3}^{n}V_{\mathrm{k}}(u,v;\mu) \tag{9}$$

where V_{k} is a polynomial in u,v of degree k and the coefficients of V_{k} satisfy

$$\chi(V)=\eta_2\,r^2+\ \eta_4\,r^4+\eta_6\,r^6+\cdots+\eta_{2i}\,r^{2i} \tag{10}$$

where $r^2=u^2+v^2$ or u^2 or v^2 or $(u^2+v^2)^2$ or other suitable forms and χ is the vector field of system (8). Here, $\eta_{2i},\ i=1,2,3,\dots$ is a polynomial in the parameter μ of the system also called the i^{th} Liapunov quantity, for more detail see ([8],[9] and [10]).
In this paper, we study the following non-linear system of differential equations

$$\begin{aligned}\dot{x}&=-z,\\ \dot{y}&=-x-z,\\ \dot{z}&=2x-\frac{13}{10}\,y+\alpha\,z+x^2+\beta\,z^2-xz,\end{aligned} \tag{11}$$

which was introduced in 2014 by Lao and Sprott, where x,y and z are variables and α,β are parameters in $\mathbb{R}$ [11]. This system is simplest electronic circuit, it is significant and used in mathematics, physics and engineering applications. Lao et al. have explained the simplest electronic circuit design. This circuit design consists of multipliers, integrator, amplifier and inverting amplifier. A new cost function base on Gaussian mixture model has been studied for parameter estimation of system (11) in [11]. Muhammed has investigated the non-integrability of system (11) and have proved that system (11) for any value of the parameters α and β has no polynomial, Darboux, rational and analytic first integrals. Also, system (11) has two exponential factors e^x and e^y with cofactors $-z$ and $-x-z$, respectively and has no invariant algebraic surfaces with nonzero cofactors. In addition, the dynamics at infinity for system (11) is analysed by using the theory of Poincaré compactification in $\mathbb{R}^3$ [12]. According to our knowledge, the Hopf bifurcation and limit cycles for the system have not studded.

In this paper, Hopf bifurcation theorem and Liapunov quantities for finding limit cycles of system (11) are used. It was shown that one stable limit cycle can be bifurcated from the origin when $\beta>\frac{1}{1274}\left(204+\sqrt{204506}\right)$ or $\beta<\frac{1}{1274}\left(204-\sqrt{204506}\right)$. Moreover, by finding Liapunov quantities, it was shown that at most two limit cycles can be bifurcated from the origin when $\beta=\frac{1}{1274}(204\pm\sqrt{204506})$. Of course, it is important to give an analytical proof for this result.

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2. STABILITY ANALYSIS:

In this section, the stability of the unique equilibrium points at the origin of system (11) is presented. It is easy to see that system (11) has a unique equilibrium $E_0 = (0,0,0)$. By linearization around E_0, the Jacobian matrix of system (11) is given by

$$J(E_0) = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 2 & -\frac{13}{10} & \alpha \end{pmatrix}$$

It's characteristic equation at E_0 is given by

$$\lambda^3 - T\lambda^2 - K\lambda - D = 0 \qquad (12)$$

where $T = \alpha$, $D = \frac{-13}{10}$,and $K = \frac{-7}{10}$. The Hurwitz matrix of the characteristic polynomial of system (11) is:

$$H_3 = \begin{pmatrix} -\alpha & 1 & 0 \\ \frac{13}{10} & \frac{7}{10} & -\alpha \\ 0 & 0 & \frac{13}{10} \end{pmatrix}$$

The principal diagonal minors are $\Delta_1 = -\alpha$, $\Delta_2 = \frac{-1}{10}(7\alpha + 13)$ and $\Delta_3 = \frac{-13}{100}(7\alpha + 13)$.

Proposition 1.

I) The equilibrium point E_0 is locally asymptotically stable if and only if $\alpha < \frac{-13}{7}$.

II) The equilibrium point E_0 is unstable if and only if $\alpha > \frac{-13}{7}$.

Proof:

I) Suppose that E_0 is locally asymptotically stable, then equation (12) has no roots with positive real parts [13]. Since $D = \frac{-13}{10} \neq 0$ and $TK + D = 0$ if and only if $\alpha = \frac{-13}{7}$, then equation (12) has no zero roots and has no pure imaginary roots for $\alpha \neq \frac{-13}{7}$, respectively. From above we can obtain that, equation (12) has no roots with zero real parts. Thus, equation (12) has all roots with negative real part when $\alpha \neq \frac{-13}{7}$, then $\Delta_1 > 0$, $\Delta_2 > 0$, and $\Delta_3 > 0$. Therefore $\alpha < \frac{-13}{7}$, is obtained.

Conversely, suppose that $\alpha < \frac{-13}{7}$, this implies that $-\alpha > 0$, $\frac{-1}{10}(7\alpha + 13) > 0$, and $\frac{-13}{100}(7\alpha + 13) > 0$, then $\Delta_1 > 0$, $\Delta_2 > 0$, and $\Delta_3 > 0$. Thus, by Routh-Hurwitz's theorem all roots of equation (12) have negative real parts [14]**.** Therefore, E_0 is locally asymptotically stable (see Fig.1a).

II) Suppose that E_0 is unstable, then at least one roots of equation (12) has positive real part [15]. When $\alpha \ngtr \frac{-13}{7}$, then $\alpha = \frac{-13}{7}$ or $\alpha < \frac{-13}{7}$.

- If $\alpha = \frac{-13}{7}$, then the roots of equation (12) are $\lambda_1 = \frac{-13}{7}$ and $\lambda_{2,3} = \pm\frac{\sqrt{70}}{10}i$ and non of them are positive, which is contradiction.

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- If $\alpha < \frac{-13}{7}$, then by Proposition 1(I), E_0 is locally asymptotically stable, which is contradiction.

Therefore $\alpha > \frac{-13}{7}$ (see Fig.1b).

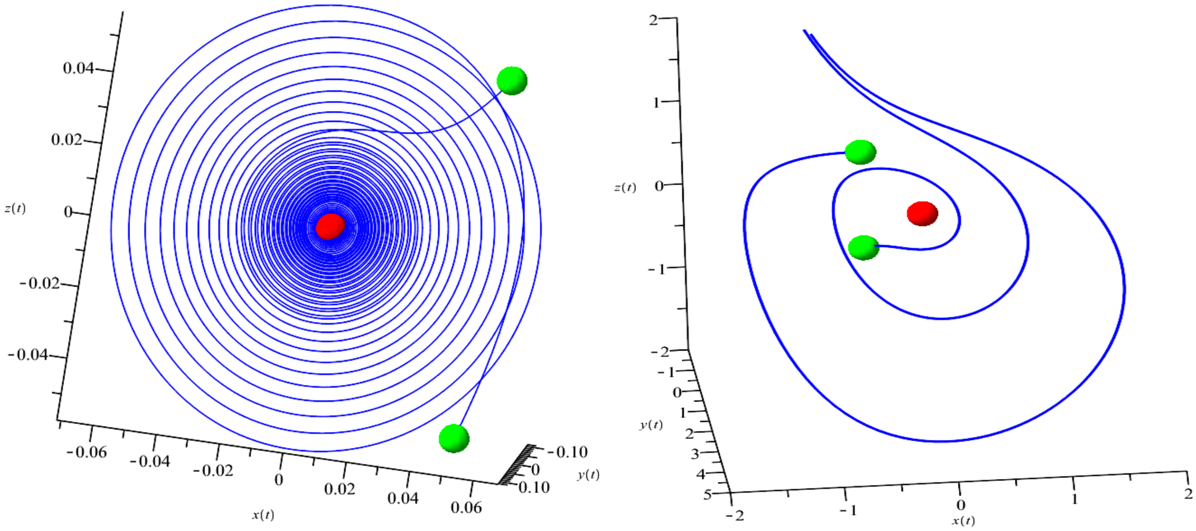

FIGURE 1. The phase portrait for system (11). (a) when $\alpha = -2$ & $\beta = 1$, E_0 is asymptotically stable. (b) when $\alpha = 2$ & $\beta = 1$, E_0 is unstable. The green and red balls indicate the initial and equilibrium points, respectively.

Conversely, suppose that $\alpha > \frac{-13}{7}$. Since D and $TK + D$ are not equal to zero, then equation (12) has no roots with zero real parts.

The roots of equation (12) are $\lambda_1 = \frac{\Omega_1}{30} - \frac{30\,\Omega_2}{\Omega_1} + \frac{\alpha}{3}$ and $\lambda_{2,3} = \left(-\frac{\Omega_1}{60} + \frac{15\,\Omega_2}{\Omega_1} + \frac{\alpha}{3}\right) \pm i\sqrt{3}\left(\frac{\Omega_1}{60} + \frac{15\,\Omega_2}{\Omega_1}\right)$.Where $\Omega_1 = \sqrt[3]{1000\,\alpha^2 - 3150\,\alpha - 17550 + 30\sqrt{\Omega_3}}$, $\Omega_2 = \frac{7}{30} - \frac{\alpha^2}{9}$ and $\Omega_3 = -3900\,\alpha^3 - 3675\,\alpha^2 + +122850\,\alpha + 352515$.

- When $\Omega_3 > 0$, then the values of $\lambda_1 = \frac{\Omega_1}{30} - \frac{30\,\Omega_2}{\Omega_1} + \frac{\alpha}{3}$ is positive, therefore E_0 is unstable (see Fig. 2 and Figure. 3a).
- When $\Omega_3 = 0$, then the values of $\lambda_1 = \frac{\sqrt[3]{1000\,\alpha^2 - 3150\,\alpha - 17550}}{30} - \frac{30\,(\frac{7}{30} - \frac{\alpha^2}{9})}{\sqrt[3]{1000\,\alpha^2 - 3150\,\alpha - 17550}}$ and it is positive at the value of vanishing Ω_3, in this case $\lambda_1 \approx 2.240$, therefore E_0 is unstable.
- When $\Omega_3 < 0$, then the real part of λ_1 is positive. Thus, E_0 is unstable (see Fig. 2 and Fig. 3b). ■

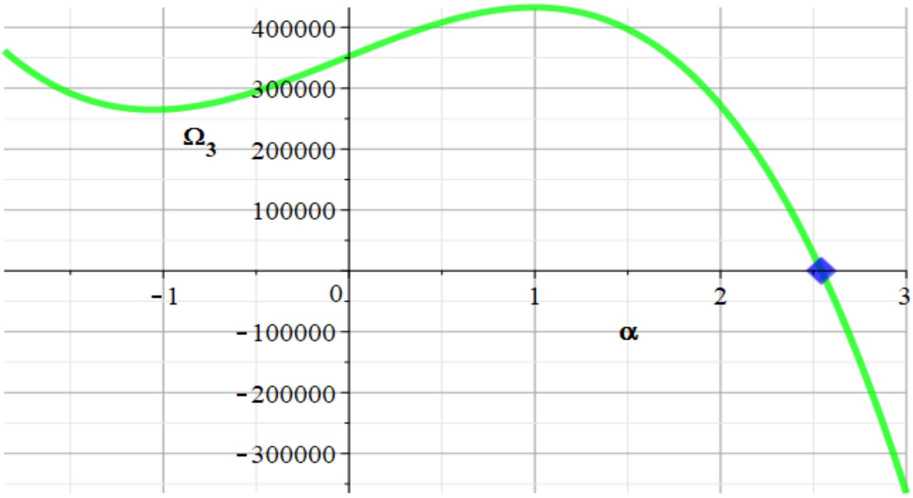

FIGURE 2: The plot of Ω_3, it is positive when $\alpha < 2.5427$, zero at $\alpha = 2.5427$ and negative when $\alpha > 2.5427$, approximately.

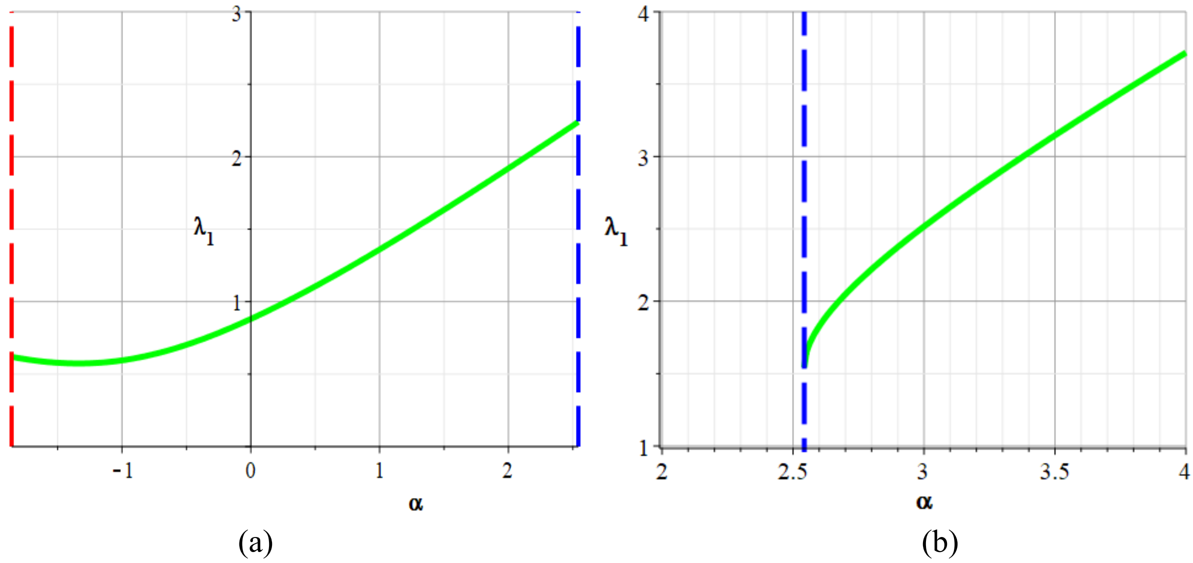

FIGURE 3. (a) The plot of eigenvalue λ_1 when $\Omega_3 > 0$ which is indicted in a green colour. (b) The plot of real part of λ_1 when $\Omega_3 < 0$, shown in a green colour. The red dash line is the value of $\alpha = -\frac{13}{7}$ and the blue dash lines indicates the value of α which vanishes Ω_3.

3. HOPF BIFURCATION ANALYSIS AND LIMIT CYCLES:

In this section, the occurrence of Hopf bifurcation and bifurcated Limit cycles of system (11) are studied.

Proposition 2. Equation (12) has two pure imaginaries with non-zero eigenvalues if and only if $\alpha = \frac{-13}{7}$. In this case the solutions of equation (12) are $\lambda_1 = -\frac{13}{7}$, $\lambda_{2,3} = \pm i\omega$ where $\omega = \frac{\sqrt{70}}{10}$.

Proof: Suppose that equation (12) has two pure imaginaries with non-zero eigenvalues. Then conditions (3) are satisfied.

From equation (12), $T = \alpha \neq 0, K = \frac{-7}{10} < 0$, $D = \frac{-13}{10} \neq 0$ and $TK + D = \frac{-7}{10}\alpha - \frac{13}{10} = 0$ at $\alpha = \frac{-13}{7}$. Therefore, $\alpha = \frac{-13}{7}$.

Conversely, suppose that $\alpha = \frac{-13}{7}$. Let λ_1 be the real solution and $\lambda_{2,3} = \pm i\omega$ be complex solutions of equation (12).

$$\lambda_1 + \lambda_2 + \lambda_3 = T \implies \lambda_1 = \frac{-13}{7} \quad \text{and} \quad \lambda_1 . \lambda_2 . \lambda_3 = D \implies \omega = \frac{\sqrt{70}}{10}.$$

Therefore, equation (12) has two pure imaginaries with nonzero eigenvalues (see Fig. 4). ■

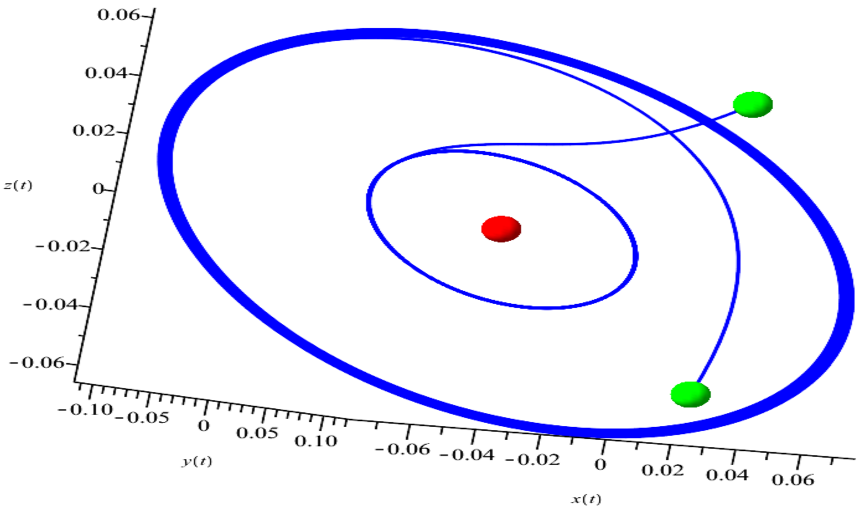

FIGURE 4: The phase portrait for system (11) when $\alpha = \frac{-13}{7}$ & $\beta = 1$, the green and red balls indicate the initial and equilibrium points, respectively.

So, the first Hopf bifurcation theorem condition is fulfilled [4].
Nevertheless, a second condition of the Hopf bifurcation theorem must be fulfilled. In the simple way:

$$\frac{d}{d\alpha} Re\left(\lambda_{2,3}(\alpha)\right) \neq 0,$$

where $Re\left(\lambda_{2,3}(\alpha)\right)$ is the real part of $\lambda_{2,3}$ which is a smooth function of α.

Proposition 3. The derivative of the real part of complex solution of the equation (12) with respect to α at $\alpha = \frac{-13}{7}$ is non-zero and equal to $\frac{343}{4066}$, i.e., $\frac{d}{d\alpha}(Re(\lambda_{2,3}(\alpha)))\Big|_{\alpha=\frac{-13}{7}} = d = \frac{343}{4066} \neq 0.$

Proof: Since $J(E_0)$ has two pure imaginary eigenvalues where $\alpha = \frac{-13}{7}$, then for α near $\frac{-13}{7}$ two of the eigenvalues will be complex conjugates. Let $\lambda_2 = u + iv$, $\lambda_3 = \overline{\lambda_2} = u - iv$ and λ_1 be eigenvalues and satisfy the following equation

$$\lambda^3 - (2u + \lambda_1)\,\lambda^2 + (|\lambda_2|^2 + 2u\,\lambda_1)\,\lambda - |\lambda_2|^2\lambda_1 = 0 \quad \text{(see [16])}$$

Equating coefficients with equation (12) results are

$$2u + \lambda_1 = \alpha,$$

$$|\lambda_2|^2\,\lambda_1 = \frac{-13}{10},$$

$$|\lambda_2|^2 + 2u\,\lambda_1 = \frac{7}{10}.$$

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Thus,

$$\frac{13}{20\,u-10\,\alpha}+2u\,(\alpha-2u)=\frac{7}{10},$$

implicitly differentiating $u=u(\alpha)$, the following is obtained:

$$u'=\frac{-20\,\alpha^2u+80\,\alpha u^2-80\,u^3-13}{20\,\alpha^3-160\,\alpha^2u+400\,\alpha u^2-320\,u^3-26},$$

at $\alpha=\frac{-13}{7}$ where $Re(\lambda_{2,3})=u=0$, we obtain:

$$u'=d=\frac{343}{4066}>0\,,where\ \alpha=\frac{-13}{7}. \qquad \blacksquare$$

So, the second condition of Hopf bifurcation theorem is fulfilled. Therefore, the Hopf bifurcation theorem holds.

Theorem 1. Under the first and second conditions of Hopf Bifurcation.

I) When $\beta>\frac{1}{1274}\left(204+\sqrt{204506}\right)$ or $\beta<\frac{1}{1274}(204-\sqrt{204506})$, the bifurcated limit cycle is unstable.
II) When $\beta\in(\frac{1}{1274}(204-\sqrt{204506})\,,\frac{1}{1274}(204+\sqrt{204506}))$, the bifurcated limit cycle is stable.

Proof: At first, we find the expression for the two-dimensional flow in the center manifold W^c at the bifurcation point.
System (11) is transformed into the Canonical form by the following linear transformation.

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=T\begin{pmatrix}u\\v\\w\end{pmatrix},where\ \ T=\begin{pmatrix}\frac{13}{20} & 1 & 1\\ 1 & 1-\frac{\sqrt{70}}{7}i & 1+\frac{\sqrt{70}}{7}i\\ \frac{169}{140} & \frac{\sqrt{70}}{10}i & \frac{-\sqrt{70}}{10}i\end{pmatrix}$$

After some calculations, the following system is obtained.

$$\begin{pmatrix}\dot u\\ \dot v\\ \dot w\end{pmatrix}=\begin{pmatrix}\frac{-13}{7}u\\ -i\frac{\sqrt{70}}{10}\,v\\ i\frac{\sqrt{70}}{10}\,w\end{pmatrix}+\begin{pmatrix}\Phi_1u^2+\Phi_2v^2+\overline{\Phi_2}\,w^2+\Phi_3\,uv+\overline{\Phi_3}\,uw+\Phi_4vw\\ \Phi_5\,u^2+\Phi_6v^2+\Phi_7\,w^2+\Phi_8\,uv+\Phi_9\,uw+\Phi_{10}\,vw\\ \overline{\Phi_5}\,u^2+\overline{\Phi_7}\,v^2+\overline{\Phi_6}\,w^2+\overline{\Phi_9}\,uv+\overline{\Phi_8}\,uw+\overline{\Phi_{10}}\,vw\end{pmatrix} \qquad (13)$$

where

$$\Phi_1=\frac{1}{28462}(28561\,\beta-7098)$$

$$\Phi_2=\frac{1}{2033}\left[\,(1400-980\,\beta)-140\sqrt{70}\,i\,\right]$$

$$\Phi_3=\frac{1}{2033}\left[\,130+338\left(\beta-\frac{7}{26}\right)\sqrt{70}\,i\,\right]$$

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$$\Phi_4 = \frac{1}{2033}(2800 + 1960\,\beta)$$
$$\Phi_5 = \frac{1}{11384800}\left[(922740 - 3712930\,\beta) + (49686 - 199927\,\beta)\sqrt{70}\; i\right]$$
$$\Phi_6 = \frac{1}{11384800}\left[(1783600\,\beta - 3508400) + (96040\,\beta + 117600)\sqrt{70}\; i\right]$$
$$\Phi_7 = \frac{1}{11384800}\left[(1783600\,\beta - 1587600) + (96040\,\beta - 392000)\sqrt{70}\; i\right]$$
$$\Phi_8 = \frac{1}{11384800}\left[(2318680\,\beta - 860860) + (152880 - 615160\,\beta)\sqrt{70}\; i\right]$$
$$\Phi_9 = \frac{1}{11384800}\left[(387660 - 2318680\,\beta) + (615160\,\beta - 178360)\sqrt{70}\; i\right]$$
$$\Phi_{10} = \frac{-1}{11384800}\left[(3567200\,\beta + 5096000) + (192080\,\beta + 274400)\sqrt{70}\; i\right]$$

According to the Center manifold theorem, the center manifold W^c is tangent to $E^c = span\{v, w\}$ at the origin [4]. Therefore, W^c can be approximated for the two variables v, w by the following equation:

$$u = h(v, w) = \sigma_1\, v^2 + \sigma_2\, vw + \sigma_3\, w^2 + O(3) \tag{14}$$

With

$$\dot{u} = \frac{\partial h}{\partial v}\,\dot{v} + \frac{\partial h}{\partial w}\,\dot{w} \tag{15}$$

It follows together with system (13) and equation (15) obtain:

$$\frac{-13}{7}\left(\sigma_1 v^2 + \sigma_2\, vw + \sigma_3\, w^2 + O(3)\right) + \Phi_1\left(\sigma_1 v^2 + \sigma_2\, vw + \sigma_3\, w^2 + O(3)\right)^2 + \Phi_2\, v^2 + \overline{\Phi_2}\, w^2 + \Phi_3\left(\sigma_1\, v^2 + \sigma_2\, vw + \sigma_3\, w^2 + O(3)\right) v + \overline{\Phi_3}\left(\sigma_1\, v^2 + \sigma_2\, vw + \sigma_3\, w^2 + O(3)\right) w + \Phi_4\, vw =$$
$$(2\sigma_1\, v + \sigma_2\, w)\left(-i\frac{\sqrt{70}}{10}\, v + \Phi_5\left(\sigma_1\, v^2 + \sigma_2\, vw + \sigma_3\, w^2 + O(3)\right)^2 + \Phi_6\, v^2 + \Phi_7\, w^2 + \Phi_8\left(\sigma_1\, v^2 + \sigma_2\, vw + \sigma_3\, w^2 + O(3)\right) v + \Phi_9\left(\sigma_1\, v^2 + \sigma_2\, vw + \sigma_3\, w^2 + O(3)\right) w + \Phi_{10} vw\right) +$$
$$(\sigma_2\, v + 2\sigma_3\, w)\left(i\frac{\sqrt{70}}{10}\, w + \overline{\Phi_5}\left(\sigma_1\, v^2 + \sigma_2\, vw + \sigma_3\, w^2 + O(3)\right)^2 + \overline{\Phi_7}\, v^2 + \overline{\Phi_6}\, w^2 + \overline{\Phi_9}\left(\sigma_1\, v^2 + \sigma_2\, vw + \sigma_3\, w^2 + O(3)\right) v + \overline{\Phi_8}\left(\sigma_1\, v^2 + \sigma_2\, vw + \sigma_3\, w^2 + O(3)\right) w + \overline{\Phi_{10}}\, vw\right) \tag{16}$$

After comparison of the coefficient for $v^2, vw, and\ w^2$ in equation (16), one can find expressions for $\sigma_1, \sigma_2\ and\ \sigma_3$.

$$\sigma_1 = \frac{1}{3112523}\left[(1117200 - 445900\,\beta) - i\left(63700\sqrt{70} + \frac{7\sqrt{70}}{5}(34300\,\beta - 49000)\right)\right]$$
$$\sigma_2 = \frac{1}{26429}(13720\,\beta + 19600)$$
$$\sigma_3 = \frac{1}{3112523}\left[(1117200 - 445900\,\beta) + i\left(63700\sqrt{70} + \frac{7\sqrt{70}}{5}(34300\,\beta - 49000)\right)\right]$$

We substitute the values of σ_1, σ_2 and σ_3 in equation (3.2) and obtain:

$$u = \frac{1}{3112523}\left[(1117200 - 445900\,\beta) - i\left(63700\sqrt{70} + \frac{7\sqrt{70}}{5}(34300\,\beta - 49000)\right)\right] v^2$$

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$$+\frac{1}{26429}(13720\,\beta+19600)\,vw+\frac{1}{3112523}[(1117200-445900\,\beta)+i\,(63700\sqrt{70}$$
$$+\frac{7\sqrt{70}}{5}(34300\,\beta-49000)\,)]\,w^2.$$

After inserting the value of u into the equations for v, w in equation (13), an approximated expression for the flow in the centre manifold is obtained:

$$\begin{pmatrix}\dot{v}\\ \\ \dot{w}\end{pmatrix}=\begin{pmatrix}-i\dfrac{\sqrt{70}}{10}\,v\\ \\ i\dfrac{\sqrt{70}}{10}\,w\end{pmatrix}+\begin{pmatrix}\Psi_1\,v^2+\Psi_2\,vw+\Psi_3\,w^2\\ \\ \overline{\Psi_3}\,v^2+\overline{\Psi_2}\,vw+\overline{\Psi_1}\,w^2\end{pmatrix}+O(3) \qquad (17)$$

Where

$$\Psi_1=\frac{7}{40660}[\,(910\,\beta-1790)+i\,(49\,\beta+60)\,\sqrt{70}\,]$$
$$\Psi_2=\frac{-7}{20330}[\,(910\,\beta+1300)+i\,(49\,\beta+70)\,\sqrt{70}\,]$$
$$\Psi_3=\frac{7}{40660}[\,(910\,\beta-810)+i\,(49\,\beta-200)\,\sqrt{70}\,]$$

By removing all the redundant non-linear terms of equation (17), the simplified expression for the flow in the centre manifold is obtained. The simplest expression is called the normal form which still contains all information about the qualitative behavior of the system at the bifurcation point. With a further linear coordinate transformation, system (17) can be rewritten into a form which is called standard form.

With

$$\begin{pmatrix}v\\ \\ w\end{pmatrix}=T\begin{pmatrix}\mathrm{p}\\ \\ \mathrm{q}\end{pmatrix},\textit{where } T=\begin{pmatrix}1 & -i\\ \\ 1 & i\end{pmatrix}$$

After some calculations, the following system is obtained.

$$\begin{pmatrix}\dot{p}\\ \\ \dot{q}\end{pmatrix}=\begin{pmatrix}-\frac{\sqrt{70}}{10}\,q\\ \\ \frac{\sqrt{70}}{10}\,p\end{pmatrix}+\begin{pmatrix}f(p,q)\\ \\ g(p,q)\end{pmatrix} \qquad (18)$$

where

$$f(p,q)=\varphi_1\,p^2-\frac{\sqrt{70}}{10}\varphi_1\,pq+\varphi_2\,q^2+\varphi_3\,p^3+\sqrt{70}\,\varphi_4\,p^2q+\varphi_5\,pq^2+\sqrt{70}\,\varphi_6\,q^3+\varphi_7\,p^4+\sqrt{70}\,\varphi_8\,p^3q+\varphi_9\,p^2q^2+\sqrt{70}\,\varphi_{10}\,p\,q^3+\varphi_{11}\,q^4.$$

$$g(p,q)=\frac{-7\sqrt{70}}{130}\varphi_1\,p^2+\frac{49}{130}\varphi_1\,pq-\frac{7\sqrt{70}}{130}\varphi_2\,q^2-\frac{7\sqrt{70}}{130}\varphi_3\,p^3-\frac{49}{13}\varphi_4\,p^2q-\frac{7\sqrt{70}}{130}\varphi_5\,pq^2-\frac{49}{13}\varphi_6\,q^3-\frac{7\sqrt{70}}{130}\varphi_7\,p^4-\frac{49}{13}\varphi_8\,p^3q+\frac{7\sqrt{70}}{130}\varphi_9\,p^2q^2-\frac{49}{13}\varphi_{10}\,p\,q^3-\frac{7\sqrt{70}}{130}\varphi_{11}\,q^4,$$

where

$$\varphi_1=\frac{-1820}{2033}$$

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$$\varphi_2 = \frac{-1274}{2033}\beta$$
$$\varphi_3 = \frac{-25480}{6327759259}(2401\beta + 15065)$$
$$\varphi_4 = \frac{-2548}{6327759259}(62426\beta^2 + 368513\beta - 104805)$$
$$\varphi_5 = \frac{178360}{6327759259}(16562\beta^2 - 7337\beta + 420)$$
$$\varphi_6 = \frac{-17836}{6327759259}(35 + 1188\beta)(26\beta - 7)$$
$$\varphi_7 = \frac{-356720}{19695296232100457}(2401\beta + 15065)^2(169\beta - 42)$$
$$\varphi_8 = \frac{9274720}{19695296232100457}(2401\beta + 15065)(49\beta - 5)(169\beta - 42)$$
$$\varphi_9 = \frac{4994080}{19695296232100457}(169\beta - 42)(4881233\beta^2 + 17567205\beta + 548400)$$
$$\varphi_{10} = \frac{64923040}{19695296232100457}(35 + 1188\beta)(49\beta - 5)(169\beta - 42)$$
$$\varphi_{11} = \frac{-17479280}{19695296232100457}(35 + 1188\beta)^2(169\beta - 42)$$

In [4], a nonlinear coordinate transformation is presented to transform every system with the following system

$$\begin{pmatrix} \dot{p} \\ \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -\omega\, q \\ \\ \omega\, p \end{pmatrix} + \begin{pmatrix} O(|p|,|q|) \\ \\ O(|p|,|q|) \end{pmatrix} \tag{19}$$

Into the system:

$$\begin{pmatrix} \dot{u} \\ \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -\omega\, v \\ \\ \omega\, u \end{pmatrix} + \begin{pmatrix} (au - bv)(u^2 + v^2) + O(4) \\ \\ (au + bv)(u^2 + v^2) + O(4) \end{pmatrix} \tag{20}$$

This is expressed in polar coordinates as:

$$\begin{pmatrix} \dot{r} \\ \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} a\, r^3 \\ \\ \omega + b\, r^2 \end{pmatrix} \tag{21}$$

At the Hopf bifurcation point, the sign of "a" determining the stability of the equilibrium point, where

$$a = \frac{1}{16}\left[f_{ppp} + f_{pqq} + g_{ppq} + g_{qqq} + \frac{1}{\omega}\left(f_{pq}\left(f_{pp} + f_{qq}\right) - g_{pq}\left(g_{pp} + g_{qq}\right) - f_{pp}\, g_{pp} + f_{qq}\, g_{qq}\right)\right] \tag{22}$$

f_{pq} denotes $\frac{\partial^2 f(0,0)}{\partial p \partial q}$,etc. and f, g are the functions containing the nonlinear terms of equation (19). Applying equation (22) to expression (18) which has structure of (19) to obtain:

$$a = \frac{1}{3112523}(436982\,\beta^2 - 139944\,\beta - 43855)$$

Fig. 5 describes the values of a and the following is obtained.

- When $\beta > \frac{1}{1274}(204 + \sqrt{204506})$ or $\beta < \frac{1}{1274}(204 - \sqrt{204506})$, then a is positive i.e. the bifurcated limit cycle is unstable and the type of Hopf bifurcation is Subcritical Hopf bifurcation.
- When $\beta \in (\frac{1}{1274}(204 - \sqrt{204506}), \frac{1}{1274}(204 + \sqrt{204506}))$, then a is negative i.e. the bifurcated limit cycle is stable and the type of Hopf bifurcation is supercritical Hopf bifurcation. ■

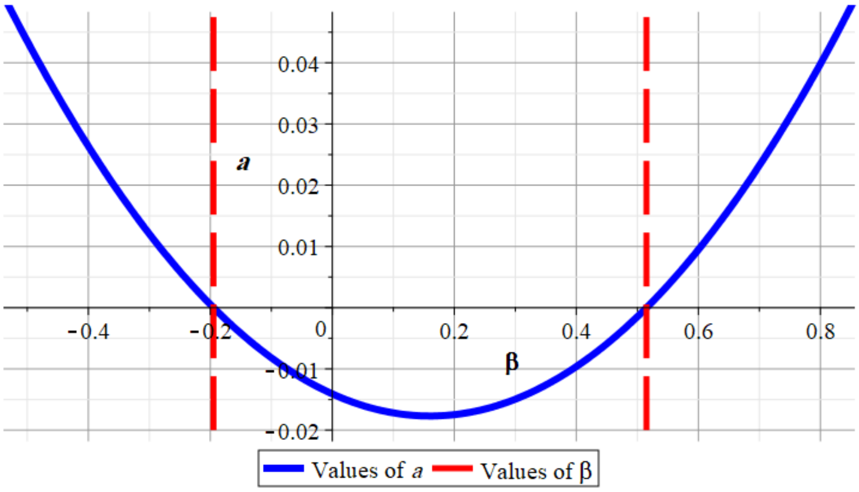

FIGURE 5: The plot of value $a = \frac{1}{3112523}(436982\,\beta^2 - 139944\,\beta - 43855)$ and the roots of a is $\beta = \frac{102}{637} \pm \frac{\sqrt{204506}}{1274}$.

4. MULTIPLE OF HOPF BIFURCATION:

The bifurcation of several limit cycles from a focus is related with the stability of the focus. Andronov have assigned a set of numbers $\eta_2, \eta_4, \eta_6, \ldots$ which they call focal values. The stability of the focus is determined by the sign of the first nonvanishing focal value. Furthermore, the number of limit cycles which may bifurcated from the focus is obtained by the number of vanishing η_i $(i = 2,4,6, \ldots)$ simultaneously [8]. For more information about this topic, the reader can refer to [17],[18], [19] and [20].

Theorem 2. When $\beta = \frac{1}{1274}(204 \pm \sqrt{204506})$, then at most two limit cycles can be bifurcated from the origin.

Proof: At first, system (11) transforms into the Canonical form by the following linear transformation.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T \begin{pmatrix} u \\ v \\ w \end{pmatrix}, where \;\; T = \begin{pmatrix} -\frac{\sqrt{70}}{7} & 1 & \frac{13}{20} \\ 0 & \frac{17}{7} & 1 \\ -\frac{\sqrt{70}}{10} & -1 & \frac{169}{140} \end{pmatrix}$$

After some calculations, the following system is obtained.

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{70}}{10}\, v \\ \frac{\sqrt{70}}{10}\, u \\ -\frac{13}{7}\, w \end{pmatrix} + \begin{pmatrix} \frac{567}{\sqrt{70}}\vartheta_1\, u^2 + 567\sqrt{70}\,\vartheta_2\, v^2 - 13689\sqrt{70}\,\vartheta_3\, w^2 - 39690\,\vartheta_4\, uv + 13689\,\vartheta_5\, uw + \frac{567}{\sqrt{70}}\,\vartheta_6\, vw \\ -140\,\vartheta_1\, u^2 - 9800\,\vartheta_2\, v^2 + 236600\,\vartheta_3\, w^2 + 140\sqrt{70}\,\vartheta_4\, uv - 338\sqrt{70}\,\vartheta_5\, uw - 140\,\vartheta_6\, vw \\ 340\,\vartheta_1\, u^2 + 23800\,\vartheta_2\, v^2 - \vartheta_3\, w^2 - 340\sqrt{70}\,\vartheta_4\, uv + \frac{57460\sqrt{70}}{7}\vartheta_5\, uw + 340\,\vartheta_6\, vw \end{pmatrix} \quad (23)$$

where

$$\vartheta_1 = \frac{1}{34561}\,(30 + 49\,\beta)$$
$$\vartheta_2 = \frac{1}{34561}\,(2 + \beta)$$
$$\vartheta_3 = \frac{1}{96770800}\,(42 - 169\,\beta)$$
$$\vartheta_4 = \frac{1}{34561}\,(23 - 14\,\beta)$$
$$\vartheta_5 = \frac{1}{691220}\,(3 - 14\,\beta)$$
$$\vartheta_6 = \frac{567}{34561\sqrt{70}}\,(52 - 169\,\beta)$$

We introduce the following Lyapunov function

$$V(u, v, w) = u^2 + v^2 + \sum_{k=3}^{n}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{i,j-i,k-j}\; u^i\, v^{j-i}\; w^{k-j} \quad (24)$$

satisfying the following equation

$$\chi(V) = \eta_2\, r^2 + \eta_4\, r^4 + \eta_6\, r^6 + \cdots + \eta_{2i}\, r^{2i} \quad (25)$$

where $r^2 = u^2 + v^2$ and χ is the vector field of system (23). By solving equation (25) and using computer algebra MAPLE, we obtain:

$\eta_2 = 0,$

$\eta_4 = \frac{1}{6225046}(1061242\,\beta^2 - 339864\,\beta - 106505)$,

$$\eta_6 = \frac{1}{1726726011260711266104}(7863141643480883032 0\,\beta^4 - 7487987307725430475 06\,\beta^3 - 99601093143458465694 8\,\beta^2 + 42719147266851158176 5\,\beta + 113442500988872872250).$$

Since $\eta_4 = 0$ if and only if $\beta = \frac{1}{1274}\left(204 \pm \sqrt{204506}\right)$. Therefore at $\beta = \frac{1}{1274}\left(204 \pm \sqrt{204506}\right)$, $\eta_4 = 0$, but $\eta_6 = \frac{1}{21728158897809}(-185734059215 \pm 356475635\sqrt{204506})$ non equal zero.

Since $\eta_2 = \eta_4 = 0$ when $\beta = \frac{1}{1274}\left(204 \pm \sqrt{204506}\right)$, then at most two limit cycles can be bifurcated from the origin.

■

5. CONCLUSIONS:

In this paper, system (11) has been investigated. The local stability and the existence of the Hopf bifurcation are studied, in addition to the direction and stability of the bifurcating periodic solutions. It was shown that the origin of system (11) is asymptotically stable and unstable when $\alpha < \frac{-13}{7}$ and $\alpha > \frac{-13}{7}$, respectively. it was proved that the bifurcated limit cycle at the bifurcation value, $\alpha = \frac{-13}{7}$, is unstable and the type of Hopf bifurcation is subcritical Hopf bifurcation when $\beta > \frac{1}{1274}\left(204 + \sqrt{204506}\right)$ or $\beta <$

$\frac{1}{1274}(204 - \sqrt{204506})$ and it is stable and the type of Hopf bifurcation is supercritical Hopf bifurcation when $\beta \in (\frac{1}{1274}(204 - \sqrt{204506}), \frac{1}{1274}(204 + \sqrt{204506}))$. Furthermore, it was also verified that at most two limit cycles can be bifurcated from the origin when $\beta = \frac{1}{1274}(204 \pm \sqrt{204506})$.

REFERENCES:

[1] Salih, R. and Hasso, M. 2017. Centre bifurcations of periodic orbits for some special three dimensional systems. *Electronic Journal of Qualitative Theory of Differential Equations*, 2017(19), 1-10.
[2] Salih, R.H., 2015. Hopf bifurcation and centre bifurcation in three dimensional Lotka-Volterra systems.
[3] Liu, W.M. 1994. Criterion of Hopf bifurcations without using eigenvalues. *Journal of Mathematical Analysis and Applications*, 182(1), 250-256.
[4] Guckenheimer, J. and Holmes, P., 2013. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields* (Vol. 42). Springer Science & Business Media.
[5] Lynch, S., 2009. *Dynamical systems with applications using MapleTM*. Springer Science & Business Media.
[6] Marsden, J.E. and McCracken, M., 2012. *The Hopf bifurcation and its applications* (Vol. 19). Springer Science & Business Media.
[7] Carr, J., 2012. *Applications of centre manifold theory* (Vol. 35). Springer Science & Business Media.
[8] Andronov, A.A., 1971. *Theory of bifurcations of dynamic systems on a plane*. Israel Program for Scientific Translations;[available from the US Department of Commerce, National Technical Information Service, Springfield, Va.].
[9] Wang, D., 1991. Mechanical manipulation for a class of differential systems. *Journal of symbolic computation*, *12*(2), pp.233-254.
[10] Lu, Z. and Luo, Y.O.N.G., 2002. Two limit cycles in three-dimensional Lotka-Volterra systems. *Computers & Mathematics with Applications*, *44*(1-2), pp.51-66.
[11] Lao, S.K., Shekofteh, Y., Jafari, S. and Sprott, J.C., 2014. Cost function based on Gaussian mixture model for parameter estimation of a chaotic circuit with a hidden attractor. *International Journal of Bifurcation and Chaos*, *24*(01), p.1450010.
[12] Muhammed S. F., 2020. *Darboux and Analytic First Integrals of a Family of a Chaotic Three Dimensional Differential Systems*, MSc. Thesis, Salahaddin University-Erbil.
[13] Pushpavanam, S., 1998. *Mathematical methods in chemical engineering*. PHI Learning Pvt. Ltd.
[14] Arrowsmith, D.K. and Place, C.M., 1982. *Ordinary Differential Equations: A qualitative approach with applications* (No. 517.38 A7).
[15] Rao, M.R.M., 1980. Ordinary differential equations, Theory and Applications, East.
[16] Merrill, S.J., 1978. A model of the stimulation of B-cells by replicating antigen—II. *Mathematical Biosciences*, *41*(1-2), pp.143-156.
[17] Buică, A., García, I.A. and Maza, S. 2012. Existence of inverse Jacobi multipliers around Hopf points in R3: emphasis on the center problem. *Journal of Differential Equations*, *252*(12), 6324-6336.
[18] Göbber, F. and Willamowski, K.D., 1979. Ljapunov approach to multiple Hopf bifurcation. *Journal of mathematical analysis and applications*, 71(2), 333-350.
[19] Lestandi, L., Bhaumik, S., Avatar, G.R.K.C., Azaiez, M. and Sengupta, T.K., 2018. Multiple Hopf bifurcations and flow dynamics inside a 2D singular lid driven cavity. *Computers & Fluids*, 166, 86-103.
[20] Rionero, S., 2019. Hopf bifurcations in dynamical systems. *Ricerche di Matematica*, 68(2), 811-840