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### **RESEARCH PAPER**

# Studying Some Stochastic Differential Equations with trigonometric terms with Application

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A B S T R A C T: In this paper we look at several (trigonometric) stochastic differential equations, we find the general form for such nonlinear stochastic differential equation by using the I'to formula. Then we find the exact solution for the different trigonometric stochastic differential equations by the use of stochastic integrals. Ilustrate the approach with various examples. (Precise solution using the Ito integral formula) and approximate solution (numerical approximation (the Euler-Maruyama technique and the Milstein method) were compared to the exact solutions with the error of those approaches

KEYWORDS: Trigonometric stochastic differential equations, Euler-Maruyama technique, Milstein method. DOI: <u>https://doi.org/10.31972/ticma22.13</u>

#### **1. INTRODUCTION:**

The most common applications are modeled using stochastic differential equations and ordinary differential equations. Many phenomena including unexpected (uncertain) consequences are typically described by adding a random or stochastic component to the ordinary differential equation, resulting in stochastic(random) differential equations (SDEs), and the word stochastic(random) is referred to as the noise term [1]. Then, an SDE is a differential equation in which one or more of the terms are stochastic (random) processes, and the solution is also stochastic as a result of the stochastic process [2] . suppose we have the following ordinary differential equation.

$$y(t) = G(t, y(t)) ; t > 0$$
 (1)

 $Y(0) = y_0$ 

If G(.) is any smooth function and y0 is any initial(fixed) point  $y0 \in Rn$ , if equation (1) contain the stochastic effects (wiener process) to describe random behavior known as Brownian motion, such equation has the form:

(3)

$$\hat{y}(t) = G(t, y(t)) + F(t, y(t))\delta(t) \quad ; \quad t > 0$$
(2)

Where  $\delta(t)$  represents the white noise (random) process (which is the formal derivative of the). Equation (2) can also be written as:

 $\frac{dy(t)}{dt} = G(t, y(t)) + F(t, y(t) \frac{dW(t)}{dt}$ \* Corresponding Author: Abdulghafoor Jasim Salim E-mail: drabdul\_salim@uomosul.edu.iq Article History: Received: 01/08/2022 Accepted: 15/09/2022 Published: 07/12/2022 Where  $\frac{dW(t)}{dt} = \delta(t)$ , dW(t) indicates the derivative of the Weiner process. multiply both side of equation (3) by dt, we get

$$dy(t) = G(t, y(t))dt + F(t, y(t)) dW(t)$$

(4)

 $y(0) = y_{0}$ 

The drift and diffusion coefficients are denoted by G(.) and F(.) correspondingly. Then we obtain a stochastic differential equations (eq.(4)).

In 2019 Junteng Jia and Austin R. Benson introduced Neural Jump Stochastic Differential Equations that provide a data-driven approach to learning continuous and discrete dynamic behavior [3], and Xuechen Li et.al generalized the adjoint sensitivity method stochastic differential equations, allowing time-efficient and constant-memory computation of gradients with high-order adaptive solvers [4], and in 2020 <u>M. M. Vas'kovskii</u> studied mixed-type stochastic differential equations driven by standard and fractional Brownian motions with Hurst indices greater than 1/3 [5], and in 2020 Saeed and Salim studied the exact and approximate solution for some harmonic stochastic differential equations, by using Ito integral formula and numerical approximation [6].

We identify the general form and investigate some product (SDE) in this article, as well as the precise solution of that equation and compare it to the exact solution of the suggested model using numerical methods.

#### **2.MAIN RESULTS:**

**Definition (1) :(**The random variable) [7]

The random variable Y(.) is a one to one function that maps from sample space  $\mu$  to the real number R (i.e. Y: $\mu \rightarrow R$ ).

#### **Definition 2:** (stochastic(random) process) [8]

A stochastic process, often known as a random process, is a mathematical object consisting of a set of random variables represented by  $\{Y(t)\}$  where t belonging to T (sorted by index set T where T  $\subseteq$  R.)

#### Definition 3: (Wiener process) [8]

 $\{W(t)\}\$  is a Wiener process (Brownian motion) over the interval [0, T] which is a continuous-time stochastic process satisfying: the following conditions: 1: W(0) = 0

- 2: suppose t,  $s \ge 0$ , then W (t)-W (s) is distributed normally, with zero mean and variance |t s|.
- 3: For  $0 \le s < t < u < v \le T$ , W(t) W(s) and W(v) W(u) are independent.

#### **3.STOCHASTIC INTEGRAL [9] [10]**

A stochastic integral is an integral which is defined as a sum more than an integration and it is increased by the rise in time on the Wiener process trajectory (Dumas and Luciano, 2017). That is :

$$\int_{a}^{b} c(t) dw(t) = \sum_{i=0}^{k-1} \delta_{i} \left( W_{t_{i+1}} - W_{t_{i}} \right)$$
(5)

Where c(t) is a real-valued stochastic process,  $c = \{c(t)\}_{a \le t < b}$  and  $\{w(t)\}$  is a Wiener process. We can write equation (4) in the integral form, that is,

$$Y(t) = y(0) + \int_0^t G(s, Y(s)) \, ds + \int_0^t F(s, Y(s)) \, dW(s) \tag{6}$$

The first integral is determistic and the second are stochastic which need much attension to be solved:

That's need The following requirements for G(s, Y(s)) and F(s, Y(s)).

$$E \int_{0}^{t} F^{2}(s, Y(s)) ds < \infty$$
$$\left| G(s, Y(s)) \right| ds < \infty$$

The stochastic integral has one of the most essential properties:

and nearly certainly for all  $t \ge 0$ 

$$\int_{0}^{t} W(S) \ dW(s) = \frac{1}{2} \int_{0}^{t} d(W^{2}(S)) - \frac{1}{2} \int_{0}^{t} ds$$

$$= \frac{1}{2} W^{2}(t) - \frac{t}{2}$$
(7)

 $\int_{0}^{t}$ 

And the general form's are: (see for example [...])

$$\int_{0}^{t} W^{m}(t) dW(t) = \frac{1}{m+1} W^{m+1}(t) + \sum_{j=0}^{m-2} (-1)^{j+1} \frac{m!}{2^{j+1}(m-j)!} W^{m-j}(t) + (-1)^{m} \frac{m!}{2^{m}} \frac{t}{2}$$
(8)

#### lemma: [11]

Consider the process  $E_t = G(t)F(W_t)$  with differentiable G and F. By using the product rule for differentiation to  $E_t$ , the following results are obtained

$$dE_t = dG(t) f(W_t) + G(t) dF(W_t)$$
  
= G'(t) F(W\_t) dt + G(t) (F'(W\_t)dWt + 1/2 F''(W\_t) dt)  
= G'(t) F(W\_t) dt + 1/2 a(t) F''(W\_t) dt + G(t) F'(W\_t)dW\_t.

Using integration by parts which is obtained by writing the connection in integral form, then we have:

$$\int_{n}^{m} G(t)F'(W_{t})d(W_{t}) = G(t)F(W_{t})\left|\frac{m}{n} - \int_{n}^{m} G'(t)F(W_{t})dt - \frac{1}{2}\int_{n}^{m} G(t)F''(W_{t})dt$$
(9)

Suppose we have a product between two functions (of t and  $W_t$ ) for which an antiderivative is known, use this formula. The two examples below are both essential and useful in our applications.

1. The aforementioned formula assumes the simple form if  $b(W_t) = W_t$ .

$$\int_{n}^{m} G(t)d(W_{t}) = G(t)W_{t} |\frac{t=m}{t=n} - \int_{n}^{m} G'(t)W_{t}dt$$
(10)

It is worth noting that the left side is a Wiener integral.

1. If a(t) = 1, then the formula becomes

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$$\int_{n}^{m} F'(W_t) d(W_t) = F(W_t) \left| \frac{t = m}{t = n} - \frac{1}{2} \int_{n}^{m} F''(Wt) dt \right|$$
(11)

Ito's integral formula: [1]

Consider Ito's stochastic differential equation, which has the following form: dy(t) = G(t, y(t))dt + F(t, y(t))dW(t)(12)

for  $0 \le t \le T$  let X(t, Y(t)) be a smooth function if we use the Taylor expansion rule, then we get:

$$dX(t,Y(t)) = \left(\frac{\partial x}{\partial t} + G(t,Y(t))\frac{\partial x}{\partial y} + \frac{1}{2}F^{2}(t,Y(t))\frac{\partial^{2}x}{\partial y^{2}}\right)dt + F(t,Y(t))\frac{\partial x}{\partial y}dW(t)$$
(13)

eq.(13) is called ito formula where y(t) is the solution of a stochastic differential equation (12). If the requisite partial derivatives are present, the mixed differentials can be joined according to the requirements.

$$dt. dW = dW. dt = (dt)^2 = 0 \& (dW)^2 = dt$$

**Proposition:** Let X(t, Y(t)) satisfies equation (14), i.e. the solution of equation (12), we can find the general form to  $X(t, Y(t)) = (Y(t))^n$ .where y(t) any trigonometric function.

i) 
$$y(t) = sin(x(t))$$
  
Here  $X(t) = U(t, sin(x(t)))$ , Then: by using *ito*-formula we have:  
 $d(X(t)) = d(sin(x(t))^{n})$   
 $= \left(n(sin(x(t))^{n-1}G(x(t), t) + \frac{1}{2}n(n-1)(sin(x(t))^{n-2}F^{2}(y(t), t))dt + (n(sin(x(t))^{n-1}F(y(t), t))dW(t) \right)$ 
(14)

we obtain the prove for equation (14) recursively:

• let m = 1

$$dX(t) = \frac{\partial U(y(t),t)}{\partial t} dt + \frac{\partial U(y(t),t)}{\partial y} dx + \frac{1}{2} F^{2}(y(t),t) \frac{\partial^{2} U(y(t),t)}{\partial y^{2}} dt$$

$$dX(t) = \frac{\partial U(y(t),t)}{\partial t} dt + \frac{\partial U(y(t),t)}{\partial y} \left( G(y(t),t) dt + F(y(t),t) dW_{(t)} \right) + \frac{1}{2} F^{2}(y(t),t) \frac{\partial^{2} U(y(t),t)}{\partial y^{2}} dt$$

$$dX(t) = \left( \frac{\partial U(y(t),t)}{\partial t} + G(y(t),t) \frac{\partial U(y(t),t)}{\partial y} + \frac{1}{2} F^{2}(y(t),t) \frac{\partial^{2} U(y(t),t)}{\partial y^{2}} \right) dt G(y(t),t) \frac{\partial U(y(t),t)}{\partial y} dW(t)$$
(15)

But, X(t) = U(y(t), t) = sin(x(t)) Then,

$$dX(t) = d(\sin(x(t))) = \cos(x(t))dx + \frac{1}{2}\sin(x(t))F^{2}(y(t),t)dt$$

$$d(\sin \sin (x(t)) = (\cos (x(t))(G(y(t), t) dt + F(y(t), t) dW(t)) + \frac{1}{2} \sin (x(t))F^{2}(y(t), t) dt$$
  
$$d(\sin \sin (x(t)) = (\cos \cos (x(t)) G(y(t), t) + \frac{1}{2 \sin \sin (x(t)) F^{2}(y(t), t)}) dt$$
  
$$+ (\cos (x(t))F(x(t), t))dW(t)$$
(16)

$$sin(t) = sin(0) + \int_{0}^{t} \left( cos \cos(s) G(y(s), s) + \frac{1}{2} sin sin(s) F^{2}(y(s), s) \right) ds + \int_{0}^{t} cos(ws) dW(s)$$
(17)

• m = 2, we have:

$$dX(t) = d(sin(x(t))^{2}$$
  
=  $(2 sin sin(x(t)) cos cos(x(t))G(y(t),t) + [sin^{2}(x(t)) - cos^{2}(x(t))]F^{2}(Y(t),t))dt$   
+  $(2 sin sin(x(t)) cos((xt))F(Y(t),t))dW(t)$ 

proof:

$$\begin{aligned} d(\sin(x(t)))^2 &= 2\sin\sin(x(t))\,\cos\cos(x(t))\,dx + [\sin^2(x(t)) - \cos^2(x(t)]F^2(y(t),t)dt \\ d(\sin(x(t)))^2 &= 2\sin\sin(x(t))\,\cos\cos(x(t))\left(G(y(t),t)dt + F(y(t),t)dW(t)\right) + [\sin^2(x(t)) \\ &- \cos^2(x(t))F^2(Y(t),t)dt \end{aligned}$$

 $d(sin(x(t)))^{2} = (2 sin sin (x(t)) cos cos (x(t)) G(y(t),t) + [sin^{2}(x(t)) - cos^{2}(x(t))]F^{2}(Y(t),t))dt$ + (2 sin sin (x(t)) cos (x(t))F(Y(t),t))dW(t)(18)

By integrating from zero to t, we get

$$sin^{2}(t) = sin^{2}(0) + \int_{0}^{t} (2 \sin \sin (s) \cos \cos (s) G(y(s), s) + [sin^{2}(s) - \cos^{2}(s)F^{2}(y(s), s))]ds + \int_{0}^{t} ((2 \sin \sin (ws) \cos (ws)F(Y(t), t)))dW(s)$$
(19)

• The solution for the general form (m = n):

$$\begin{aligned} X(t) &= d(sin(x(t)))^n \\ &= \left( n \, (sin \, (x(t)))^{n-1} G(y(t), t) + \frac{1}{2} n \, (n-1) (sin \, (x(t)))^{n-2} F^2(y(t), t) \right) dt \\ &+ (n \, (sin \, (x(t)))^{n-1} F(y(t), t) dW(t) \end{aligned}$$

proof:

$$d(\sin(x(t)))^{n} = n(\sin(x(t)))^{n-1}dx + \frac{1}{2}n(n-1)(\sin(x(t)))^{n-2}F^{2}(y(t),t)dt$$

$$d(\sin(x(t))^{n} = n(\sin(x(t)))^{n-1} (G(t, y(t))dt + F(t, y(t))dW(t)) + \frac{1}{2}n(n - 1)(\sin(x(t)))^{n-2} F^{2}(y(t), t)dt$$

$$d(sin(x(t)))^{n} = \left(n(sin(x(t)))^{n-1}G(y(t),t) + \frac{1}{2}n(n-1)(sin(x(t)))^{n-2}F^{2}(y(t),t)\right)dt + (n(sin(x(t)))^{n-1}F(y(t),t))dW(t)$$
(20)

By integrating from zero to t, we get

$$sin(t) = sin(0) + \int_{0}^{t} \left( n(sin(s))^{n-1} G(y(s), s) + \frac{1}{2} n(n-1)(sin(s))^{n-2} F^{2}(y(s), s) \right) ds + \int_{0}^{t} \left( n(sin(s))^{n-1} F(y(s), s) \right) dW(s)$$
(21)

#### ii) suppose that Y(t) = cos(x(t)), then as before we obtain

$$\begin{aligned} X(t) &= d(\cos(x(t)))^n \\ &= \left( n(\cos(x(t)))^{n-1} G(y(t), t) + \frac{1}{2} n(n-1) (\cos(x(t)))^{n-2} F^2(y(t), t) \right) dt \\ &+ (n(\cos(x(t)))^{n-1} F(y(t), t)) dW(t) \end{aligned}$$
(22)

By integrating from zero to t, we get

$$cos(t) = cos(0) + \int_{0}^{t} \left( n(cos(s))^{n-1} G(y(s), s) + \frac{1}{2}n(n-1)(cos(s))^{n-2} F^{2}(y(s), s) \right) ds + \int_{0}^{t} \left( n(cos(s))^{n-1} F(y(s), s) \right) dW(s)$$
(23)

iii) Y(t) = tan(x(t)), then:

$$\begin{aligned} X(t) &= d(tan(x(t)))^n \\ &= \left( n(tan(x(t)))^{n-1} G(y(t), t) + \frac{1}{2} n(n-1) (tan(x(t)))^{n-2} F^2(y(t), t) \right) dt \\ &+ (n(tan(x(t)))^{n-1} F(y(t), t)) dW(t) \end{aligned}$$
(24)

By integrating from zero to t, we get

$$tan(t) = tan(0) + \int_{0}^{t} \left( n(tan(s))^{n-1} G(y(s), s) + \frac{1}{2}n(n-1)(tan(s))^{n-2} F^{2}(y(s), s) \right) ds + \int_{0}^{t} \left( n(tan(s))^{n-1} F(y(s), s) \right) dW(s)$$
(25)

#### **EXAMPLES:**

In this paragraph we explain the method by introducing some examples. Let equation (12) be given. i. e.

$$dY(t) = G(t, y(t))dt + F(t, y(t))dW(t)$$

#### Example 1:

find the exact solution for equation (16), where G = 0 and F = 1, X(t) = t, then dY(t) = dt + dW(t). and let the initial condition X(0) = 0.1From equation (18) we get

$$d(\sin\sin(x(t))) = \left(-\frac{1}{2}\sin(x(t))\right)dt + \cos(x(t))dW(t)$$
(26)

and therefore, the exact solution is found:

$$sin(t) = sin sin (0) - sin sin (W_t - W0)$$
(27)

#### Example 2:

find the exact solution for equation (18), where G = 1 and F = 1, X(t) = t, then dY(t) = dt + dW(t). and also let the initial condition x(0) = 0.1

From equation (18) we get.

$$d(\sin(t))^{2} = (2 \sin \sin (t) \cos \cos (t) - \sin^{2}(t) + \cos^{2}(t))dt + 2 \sin \sin (t)$$
  
$$\cos \cos (t) dW(t) \quad (28)$$

and therefore, the exact solution is found:

$$\sin^{2}(t) = \sin(t)\cos(t) + \frac{1}{2}\exp(2t) - \frac{1}{2}\cos(2W_{t})$$
<sup>(29)</sup>

#### **4. NUMERICAL SOLUTION:**

We employ numerical approximations such as the Euler-Maruyama and Milstein methods, as well as calculating the error to explain solution convergence (exact and approximation).

Suppose we have the following stochastic differential equation:

$$dY(t) = G_n(t, Y(t))dt + F_n(t, Y(t))dW(t) , \qquad y(t0) = x0$$

#### Euler Maruyama method has the form [15] :-

$$y_{t_{n+1}} = y_{t_n} + G_n(t_{n+1} - t_n) + F_n(W_{t_{n+1}} - W_{t_n})$$
(30)  

$$Y_{t_{i+1}} = Y_{t_i} + G_n\Delta t + F_n\sqrt{\Delta t}\eta_i$$
(31)  
With  $y(0) = 0.1$  and  $t_i = i\Delta t \Rightarrow \Delta t = \frac{1}{N}, \ \eta i \sim (0,1), t \in [0,1]$   
Example 1 is a the difference of the difference of

Equation (31) is called Euler-Maruyama method.

#### **5.MILSTEIN'S METHOD HAS THE FORM [15] :**

In the 1-dimensional case with d = m = 1, we add to the Euler scheme the term  $\frac{1}{2}F_{y_t}[(dW_t)^2 - dt]$ Therefore, we have from the Ito-Taylor expansion the Milstein method given below

$$y_{t_{n+1}} = y_{t_n} + G_n(t_{n+1} - t_n) + F_n(W_{t_{n+1}} - W_{t_n}) + \frac{1}{2}F_nF_{x_n}[(dW_t)^2 - dt]$$
(32)

In the multi-dimension case with m = 1 and  $d \ge 1$  the *j* th component of the Milstein method is given by :

$$y_{t_{n+1}}^{j} = y_{t_{n}}^{j} + G_{n}^{j}(t_{n+1} - t_{n}) + F_{n}^{j}(W_{t_{n+1}} - W_{t_{n}}) + \frac{1}{2}\sum_{i=1}^{d} F_{n}^{j}F_{x_{n}}^{j}[(dW_{t})^{2} - dt]$$
(33)

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$$y_{t_{i+1}} = y_{t_i} + \Delta t + \sqrt{\Delta t}\eta_i + \frac{1}{4}(\eta_i^2 - 1)\Delta t$$
(34)

#### 6. Results:

1: Suppose we have equation (26), that is

$$d(\sin \sin (y(t))) = \left(-\frac{1}{2}\sin(y(t))\right)dt + \cos \cos (y(t)) dW(t)$$
(26)  
and therefore, the exact solution is found:

 $\sin(t) = \sin \sin(0) - \sin \sin(W_t - W_0)$ (27)

The following graphs were obtained from the *Matlab*<sup>™</sup> program implementing the algorithm: see [9]



Figure 1: Exact solution charts vs numerical solutions produced using the Euler Maruyama method. With N= $2^8 \& R=2$ 



Figure 2 shows a comparison of precise answers to numerical solutions produced using Milstein's technique. With  $N=2^8 \& R=2$ 

Error	Ν	R	numerical method
0.0040	2 <sup>8</sup>	2	Euler Maruyama method
0.0846	2 <sup>8</sup>	2	Milstein's method

Table1 shows the comparison of the Euler-Maruyama and Millstein's Method outcomes.

The numerical solution is relatively near to the precise answer, and Euler-Maruyama is more accurate than Millstein's Method, as seen in the above figures and table.

2: The following graphs are for example two equations (28)



Figure 3: Comparison of precise and numerical solutions derived using the Euler Maruyama technique. With N=2^8 & R=2





Table 2: Analysis of the Euler-Maruyama and Millstein's Method outcomes in comparison.

Error	Ν	R	numerical method
0.0023	2 <sup>8</sup>	2	Euler Maruyama method
0.0568	2 <sup>8</sup>	2	Milstein's method

The error between the Euler Maruyama approach and Milstein's method is explained in the table above (See for example [15]).

We can also see that the Euler Maruyama method is closer to the Exact Solution than the Milstein method when  $N = 2^8$ .

#### 6. CONCLUSION:

In this paper, we find the general form of a stochastic differential equation with limits of trigonometric functions using the ito formula and its solutions, and we have applied those results by using some examples to explain the methods and use one of the simplest numerical methods (Euler-Maruyama and Milstein methods) to find approximate solutions. We see that Euler-Maruyama is better than the Milstein method's convergence of micro-stochasticity.

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