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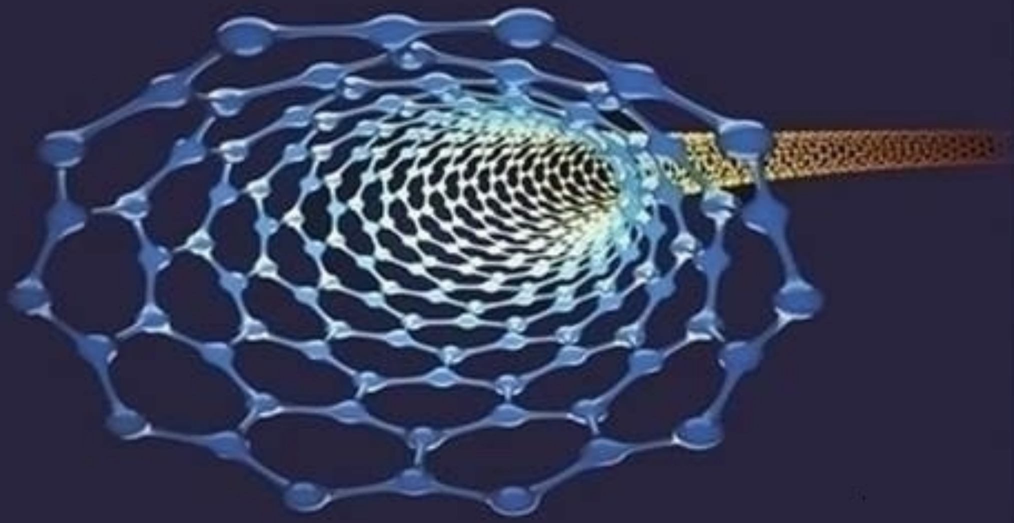

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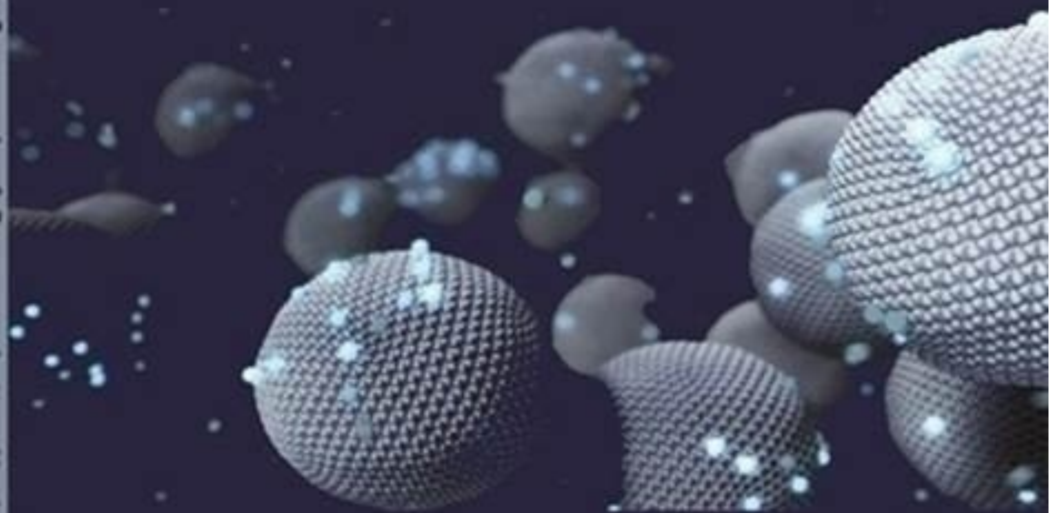

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**Proceeding of 3rd International Conference of Mathematics and its Applications TICMA2022**
https://conferences.su.edu.krd/su/ticma2022/
**DOI: https://doi.org/10.31972/ticma2022**

RESEARCH PAPER

# Combinations of $\boldsymbol{L}$-Complex Fuzzy $\boldsymbol{t}$-Norms and $\boldsymbol{t}$-Conorms

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**A B S T R A C T:**
This paper investigates the study of $L$-complex fuzzy sets. The $L$-complex fuzzy set, where $L$ is a completely distributive lattice, is a generalization of the complex fuzzy set. The fundamental set theoretic operations on $L$-complex fuzzy sets are discussed properly, including $L$-complex fuzzy complement, union and intersection. New procedures are presented to combine the novel concepts of $L$-complex fuzzy $t$-norms and $t$-conorms and look into the conditions that lead to a comparable representation theorem. We have used the axiomatic method, in the sense that our underlying assumptions, especially about $L$, are abstract; it can thus be ascertained to what extent our results apply to some new problem. On the other hand, our method shows that if mathematics, as we use it, is consistent, so is fuzziness, as we formulate it.

KEY WORDS: Fuzzy set; Complex fuzzy set; $L$-complex fuzzy set; Fuzzy $t$-norm; Fuzzy $t$-conorm.
DOI: https://doi.org/10.31972/ticma22.16

## 1. INTRODUCTION:

Zadeh developed the notion of fuzzy sets in [1]. Many basic ideas held by scholars at the time were called into question by this unique way of portraying uncertainty. Fuzzy sets, in particular, created a problem for probability theory. Proposing a new instrument for modeling uncertainty, and its core foundation: Aristotelian two-valued logic [2]. As a result, the new theory came under attack. On the other hand, fuzzy sets have grown in popularity in recent years. Have proved to be a popular approach for portraying uncertainty. They're beneficial in a variety of situations. A few decades have passed following Zadeh's landmark paper. There has been a lot of research into fuzzy sets.

Ramot *et al*. [3] were among the first to propose the concept of a complex fuzzy set. As a result, a complex fuzzy set has a range that extends from the closed interval $[0, 1]$ to a disk of radius one on the complex plane. They introduce set-theoretic operations on complex fuzzy sets such as, intersection, union, complement, rotation, and reflection.

Also, provides De Morgan's laws for complex fuzzy sets and complex fuzzy relationships. The phase component of complex fuzzy set memberships, according to Ramot *et al*., may be used to depict periodic issues or recurrent problems considerably more accurately, such as reflecting the influence of financial indicators from two nations on each other over time. He suggested that signal processing is another sector where a complex fuzzy set may be useful. Furthermore, one of the desired uses of complex fuzzy sets, according to Dick [4], is to depict events with roughly periodic activity.

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**Article History:**
Received: 01/08/2022
Accepted: 15/09/2022
Published: 07/12/2022

DOI: https://doi.org/10.31972/ticma2022

Congestion in a large metropolis is a periodic phenomenon that never repeats itself. As a result, sophisticated fuzzy logic, rather simply fuzzy logic, can be utilized to answer certain types of problems more efficiently and precisely.

The significance of this work may lie more in its point of view than in any particular results. The theory is still young, and no doubt many concepts have yet to be formulated, while others have yet to take their final form. However, it should now be possible to visualize the outlines of the theory. One reason for this is the natural feeling that probability theory is not appropriate for treating the kind of uncertainty that appears in study fuzzy systems; this uncertainty seems to be more of an ambiguity than a statistical variation. Similar difficulties arise in a wide variety of problems. It is characteristic of attempts to apply probability theory to them that it is difficult or impossible to estimate the distributions assumed to be involved, that there is uncertainty about the nature of the statistical independence, or that certain parameters are ignored, taken as given, or found difficult to estimate. Under these circumstances, the chief use of $L$ theory has been to partially justify intuitively appealing procedures, to suggest procedures already found useful in complex fuzzy sets, or to provide some sort of insight into the nature of things. This paper develops a basic language, and some combination properties, mainly formal and algebraic, and prepares for new points of view. However, there are related topics of greater mathematical depth. These include an information theory for complex fuzzy sets, the fuzzification of various mathematical structures, and a more detailed treatment of the lattice problem.

The paper has been divided into three sections: In section two, we go through the topic's definitions and ideas, as well as the notations that will be used in the rest of the paper. In section three, we generalize the complex fuzzy set to the $L$-complex fuzzy set where $L$ is a completely distributive lattice. In addition, we describe various additional operations and laws for an $L$-complex fuzzy concerning the $L$-complex fuzzy union, intersection, and complement function, such as distributive property, idempotent property, absorption rule, and so on. In section four, we offer some fundamental $L$-complex fuzzy set findings for $L$-complex fuzzy union, $L$-complex fuzzy intersection, and $L$-complex fuzzy complement and examine specific instances of these. The last section contains our key results conclusions.

## 2. PRELIMINARIES

Fuzzy subsets are defined by their membership function but their values can be any number in the unit interval [0,1]. We will use the notation of placing a "bar" over a letter to denote a fuzzy subset. As a consequence, when we replace the requirement that membership functions must be in the unit interval [0, 1] to represented by symbols of an arbitrary set $L$, that is, at least partially ordered, we obtain fuzzy sets of another generalized type. They are called $L$-fuzzy sets [5], and their membership grades are from universal set $X$ into $L$ (set $L$ with its ordering is most frequently a lattice).

To study the algebra of fuzzy sets of $X$, we have to specify union and intersection. Let $\tilde{x} = \tilde{x}_1 \cup \tilde{x}_2$ or $\tilde{x} = \tilde{x}_1 \cap \tilde{x}_2$. The value of $\tilde{x}(x)$ will be a function of the two values $\tilde{x}_1(x)$ and $\tilde{x}_2(x)$ so that $\tilde{x}(x) = u(a, b)$ for union and $\tilde{x}(x) = i(a, b)$ for intersection, with $a = \tilde{x}_1(x)$, $b = \tilde{x}_2(x)$ in the interval [0,1].

Choices for $u$ could be $u(a, b) = a + b - ab$, $u(a, b) = \max(a, b)$, and $u(a, b) = \min\left(1, \sqrt{a^2 + b^2}\right)$. Choices for $i$ are $i(a, b) = ab$, $i(a, b) = \min(a, b)$, and $i(a, b) = \sqrt{\max(0, a^2 + b^2 - 1)}$.

A function $C: [0,1] \times [0,1] \to [0,1]$ is a $t$-conorm satisfying the following four axioms:

(1) $C(a, 0) = a$,
(2) $C(a, b) = C(b, a)$,
(3) $C(a, C(b, c)) = C(C(a, b), c)$,
(4) $a \leq b$ and $c \leq d$ implies $C(a, c) \leq C(b, d)$.

Axiom 1 is a boundary condition that implies $C(1,1) = 1$, $C(1,0) = 1$. Axiom 2 is symmetric condition and we see that $C(0,1) = 1$ too. Axiom 3 and 4 then says that $C$ is associative and $0 = C(0,0) \leq C(0,1)$.

A $t$-norm $T$ is a function $y = T(a, b)$, $y, a, b \in [0,1]$ satisfying the following axioms:

(1) $T(a, 1) = a$,

**DOI: https://doi.org/10.31972/ticma2022**

(2) $T(a,b) = T(b,a)$,
(3) $T(a,T(b,c)) = T(T(a,b),c)$,
(4) $a \leq b$ and $c \leq d$ implies $T(a,c) \leq T(b,d)$.

Axioms 1, 2 and 4 give $T(1,1) = 1$, $T(0,1) = T(1,0) = T(0,0) = 0$. In sum, $T$ is associative (Axiom 3) which we will need in later sections. section may be divided by subheadings. It should provide a concise and precise description of the experimental results, their interpretation, as well as the experimental conclusions that can be drawn.

## 3. OPERATIONS OF *L*-COMPLEX FUZZY SETS

The following $L$-Complex fuzzy complement intersection, and union special operations are utilized in this section.

### 3.1. *L*-Complex Fuzzy Complement

Let $\tilde{Z}$ be a $L$-complex fuzzy set on a universal set $S$ and let $\mu(\tilde{Z}|x)$ be $x$'s complex grade of membership in $\tilde{Z}$. Let $C\tilde{Z}$ denote the $L$-complex fuzzy complement of $\tilde{Z}$ of type $C$, defined by the function $C\colon \hat{z}_1 \coloneqq \{z_1 | z_1 \in \mathbb{C}, |z_1| < k, k = supL\} \longrightarrow \hat{z}_2 \coloneqq \{z_2 | z_2 \in \mathbb{C}, |z_2| < k, k = supL\}$, which assigns $C\left(\mu(\tilde{Z}|x)\right) = C\mu(\tilde{Z}|x)$, for all $x \in S$. The $L$-complex fuzzy complement, function $C$ must satisfy at least two of the following requirements:

(1) $|z_1| = 0$ implies $|C(z_1)| = 1$ and $|z_1| = 1$ implies $|C(z_1)| = 0$.
(2) For all $z_1, z_2 \in \hat{z}$ if $|z_1| \leq |z_2|$, then $|C(z_1)| \geq |C(z_2)|$.

In most cases, $C$ should satisfy various additional requirements:

(3) $C$ is a continuous function,
(4) For all $z_1 \in \hat{z}$, then $C(C(z_1)) = z_1$.

**Theorem 3.1.1.** For all $z \in \hat{z}$ and $\alpha > 0$ The function $|C(z)| = \frac{\alpha^2(1-|z|)}{|z|+\alpha^2(1-|z|)}$ is an $L$-complex fuzzy complement.

**Proof**: To prove that function is an $L$-complex fuzzy complement, we have to show that it satisfies properties (1) and (2) of $L$-complex fuzzy complement $C$.

(1) For any $z \in \hat{z}$, if $|z| = 0$, then $|C(z)| = \frac{\alpha^2(1-|z|)}{|z|+\alpha^2(1-|z|)} = \frac{\alpha^2(1-0)}{0+\alpha^2(1-0)} = 1$, and let $|z| = 1$, then clear that

$$|C(z)| = \frac{\alpha^2(1-|z|)}{|z|+\alpha^2(1-|z|)} = \frac{\alpha^2(1-1)}{1+\alpha^2(1-1)} = 0.$$

Hence, $|C(z)| = \frac{\alpha^2(1-|z|)}{|z|+\alpha^2(1-|z|)}$ is as properties (1).

(2) For all $z_1, z_2 \in \hat{z}$, if $|z_1| \leq |z_2|$, then

$$|C(z_1)| = \frac{\alpha^2(1-|z_1|)}{|z_1|+\alpha^2(1-|z_1|)} \geq \frac{\alpha^2(1-|z_2|)}{|z_2|+\alpha^2(1-|z_2|)} = |C(z_2)|$$

Hence, $|C(z)| = \frac{\alpha^2(1-|z|)}{|z|+\alpha^2(1-|z|)}$ satisfies properties (2). So that it is a $L$-fuzzy complexities complement.

**Definition 3.1.2**. For any $L$-complex fuzzy complement $C$, we say $z_e \in \hat{z}$ is the equilibrium of $C$, if $|C(z_e)| = |z_e|$.

**Theorem 3.1.3.** For any $L$-complex fuzzy complement $C$, the absolute value of equilibrium of $C$ is unique.

**Proof:** Suppose that $z_{e_1}$ and $z_{e_2}$ are any two equilibrium of $L$-complex fuzzy complement of $C$, such that $|z_{e_1}| < |z_{e_2}|$. Then by Definition 3.1.2, we conclude that $|C(z_{e_1})| - |z_{e_1}| = 0$ and $|C(z_{e_2})| - |z_{e_2}| = 0$, that is $|C(z_{e_1})| - |z_{e_1}| = |C(z_{e_2})| - |z_{e_2}|$, and by axiom (2) we now $C$ is non increasing, this means that

**DOI: https://doi.org/10.31972/ticma2022**

$\left|C\left(z_{e_1}\right)\right| \geq \left|C\left(z_{e_2}\right)\right|$, this implies that $\left|C\left(z_{e_1}\right)\right| - \left|z_{e_1}\right| > \left|C\left(z_{e_2}\right)\right| - \left|z_{e_2}\right|$, which is a contradiction of our assumption.

**Theorem 3.1.4.** Suppose that $C$, is any $L$-complex fuzzy complement and has an equilibrium $z_e$, then

$$|z| \leq |C(z)| \text{ if and only if } |z| \leq |z_e|, \qquad |z| \geq |C(z)| \text{ if and only if } |z| \geq |z_e|$$

for all $z \in \hat{z}$.

**Proof**: Suppose $|z| > |z_e|$, $|z| = |z_e|$ and $|z| < |z_e|$ in turn. Then, since $C$ is non increasing, so that by (2), $|C(z)| \leq |C(z_e)|$, for $|z| > |z_e|$, $|C(z)| = |C(z_e)|$, for $|z| = |z_e|$ and $|C(z)| \geq |C(z_e)|$, for $|z| < |z_e|$. Since clearly by definition equilibrium $|C(z_e)| = |z_e|$. So that, by substitution, we can say $|C(z)| \leq |z_e|$, $|C(z)| = \left||z_e|\right|$ and $|C(z)| \geq \left||z_e|\right|$, respectively. From our assumption we can additional rewrite these as $|C(z)| > |z|$, $|C(z)| = |z|$, and $|C(z)| < |z|$, respectively. So that if $|z| \leq |z_e|$ by above inequality ( $|C(z)| \geq |z_e|$ ), we get $|z| \leq |C(z)|$, also when, $|z| \geq |z_e|$, by inequality $|C(z)| \leq |z_e|$, we get $|z| \geq C(z)$. Conversely, similar form.

**Definition 3.1.5**. For each $L$-complex fuzzy complement $C$, we called $z_d \in \hat{z}$ is dual point of $z \in \hat{z}$, such that $|C(z_d)| - |z_d| = |z| - |C(z)|$.

**Theorem 3.1.6**. If $z_e$ is the equilibrium of complement $C$, then $(z_e)_d = z_e$.
**Proof:** If we assume that $z_e = z \in \hat{z}$ then by Definition of equilibrium $|C(z_e)| - |z_e| = 0$. Moreover, if $z_e = z_d$, then $|C(z_d)| - |z_d| = 0$. So, it is clear that the equation $|C(z^d)| - |z^d| = |z| - |C(z)|$ holds when $z_d = z_e = z$.

**Theorem 3.1.7**. For all $z \in \hat{z}$, $z_d = C(z)$ if and only if $\left|C\left(C(z)\right)\right| = |z|$.
**Proof**: Suppose that $z_d = C(z)$, By Definition 3.1.5, we get $\left|C\left(C(z)\right)\right| - |C(z)| = |z| - |C(z)|$. So that $\left|C\left(C(z)\right)\right| = |z|$. Conversely, let $\left|C\left(C(z)\right)\right| = |z|$, then by substation $\left|C\left(C(z)\right)\right|$ for $|z|$, in Definition 3.1.5, we get $|C(z_d)| - |z_d| = \left|C\left(C(z)\right)\right| - |C(z)|$, This implies that $z_d = C(z)$.

### 3.2 *L*-Complex Fuzzy Intersection *t*-Norms

Let $\tilde{Z}_1$ and $\tilde{Z}_2$ be any two $L$-complex fuzzy sets with complex-valued membership functions $\mu(\tilde{Z}_1|x)$ and $\mu(\tilde{Z}_2|x)$, respectively. The intersection between them is defined by a function of form $I: \hat{z}_1 \times \hat{z}_1 \longrightarrow \hat{z}_2$, which returns the membership function of $\tilde{Z}_1 \cap \tilde{Z}_1$ assigns $\mu\left(\tilde{Z}_1 \cap \tilde{Z}_1 \middle| x\right) = I\left(\mu\left(\tilde{Z}_1 \middle| x\right), \mu(\tilde{Z}_2|x)\right)$, for all $x \in S$. A fuzzy intersection $I$ is a binary operation satisfied that s at least four of the following properties for all $z_1, z_2, z_3, z_4 \in \hat{z}\}$:

(1) If $|z_2| = 1$, then $|I(z_1, z_2)| = |z_1|$;
(2) $|z_2| \leq |z_3|$ implies $|I(z_1, z_2)| \leq |I(z_1, z_3)|$;
(3) $I(z_1, z_2) = I(z_2, z_1)$;
(4) $I\left(z_1, I(z_2, z_4)\right) = I(I(z_1, z_2), z_4)$.

In most cases, $I$ should satisfy various additional requirements:

(5) $I$ is a continuous function;
(6) $|I(z_1, z_1)| < |z_1|$;
(7) $|z_1| \leq |z_3|$ and $|z_2| \leq |z_4| \Longrightarrow |I(z_1, z_2)| \leq |I(z_3, z_4)|$.

We have the following examples that can be used as an $L$-complex fuzzy intersection, each defined for all $z_1, z_2 \in \hat{z}$,

(1) $I_m(z_1, z_2) = \min(|z_1|, |z_2|)$.
(2) $I_P(z_1, z_2) = |z_1| \cdot |z_2|$.
(3) $I_b(z_1, z_2) = \max\{|z_1| + |z_2| - 1, 0\}$.

**DOI: https://doi.org/10.31972/ticma2022**

(4) $I_d(z_1, z_2) = \begin{cases} |z_1| & \text{when } |z_2| = 1 \\ |z_2| & \text{when } |z_1| = 1. \\ 0 & \text{otherwIse} \end{cases}$

**Proposition 3.2.1.** For all $z_1, z_2 \in \hat{z}$, $I_d(z_1, z_2) \leq |I(z_1, z_2)| \leq I_m(z_1, z_2)$.
Proof: First we prove the left-hand right inequality, if $|z_2| = 1$ then by boundary condition $|I(z_1, z_2)| = |z_1|$ and if $|z_1| = 1$ by the same condition, we get $|I(z_1, z_2)| = |z_2|$. On the other hand, we have $|I(z_1, z_2)| \leq \min(|z_1|, |z_2|)$, so that $|I(z_1, 0)| \leq min(|z_1|, 0) = 0$ and $|I(0, z_1)| \leq min(0, |z_1|) = 0$. This is enough to say $|I(z_1, 0)| = |I(0, z_2)| = 0$. Therefore, by monotonicity condition we obtain that $|I(z_1, z_2)| \geq |I(z_1, 0)| = |I(0, z_2)| = 0$.
Now, we prove of the second inequality. By the monotonicity condition we have $|I(z_1, z_2)| \leq |I(z_1, 1)|$, and by boundary condition we have to get $|I(z_1, z_2)| \leq |I(z_1, 1)| = |z_1|$. Next by commutatively $|I(z_1, z_2)| = |I(z_2, z_1)| \leq |I(z_2, 1)| = |z_2|$. Hence, $|I(z_1, z_2)|$ is less than or equal to $|z_1|$ and $|z_2|$, and this implies that $|I(z_1, z_2)| \leq min(|z_1|, |z_2|)$.

**Proposition 3.2.2.** For all $z_1, z_2 \in \hat{z}$, $I_p(z_1, z_2) \leq I_m(z_1, z_2)$.
**Proof**: At first, if $|z_1| = 0$ or $|z_2| = 0$, then $I_p(z_1, z_2) = I_m(z_1, z_2) = 0$. Next, if $|z_1| = 1$ (resp. $|z_2| = 1$), then $I_p(z_1, z_2) = I_m(z_1, z_2) = |z_2|$ (resp. $I_p(z_1, z_2) = I_m(z_1, z_2) = |z_1|$). Finally, if $(|z_1|, |z_2| < k, k = supL)\backslash\{0,1\}$, then we obtain that $|z_1| \cdot |z_2| \leq |z_1|$ and $|z_1| \cdot |z_2| \leq |z_2|$. In other words, $|z_1| \cdot |z_2| \leq \min(|z_1|, |z_2|)$, implies that $I_p(z_1, z_2) \leq I_m(z_1, z_2)$.

**Proposition 3.2.3.** For all $z_1, z_2 \in \hat{z}$, $I_b(z_1, z_2) \leq I_p(z_1, z_2)$.
**Proof:** In the case of $|z_2|$ equals one or zero, then $I_b(z_1, z_2) = I_p(z_1, z_2)$. But if $(|z_1|, |z_2| < k, k = supL)\backslash \{0,1\}$, then $I_b(z_1, z_2) = \max\{0, |z_1| + |z_2| - 1\} \leq I_p(z_1, z_2)$ because of $|z_1| + |z_2| - 1 \leq |z_1| \cdot |z_2|$.

**Proposition 3.2.4.** For all $z_1, z_2 \in \hat{z}$, $I_d(z_1, z_2) \leq I_b(z_1, z_2)$.
**Proof:** First, if $|z_1| = 1$ (resp. $|z_2| = 1$), then $I_d(z_1, z_2) = |z_2| = I_b(z_1, z_2)$ (resp. $I_d(z_1, z_2) = |z_1| = I_b(z_1, z_2)$). Next, if at least one of $|z_1|$, $|z_2|$ is zero, $\forall z_1, z_2 \in \hat{z}$ then $I_b(z_1, z_2) = 0 = I_d(z_1, z_2)$. Finally, if $(|z_1|, |z_2| < k, k = supL)\backslash\{0,1\}$, then $I_b(z_1, z_2) = \max\{0, |z_1| + |z_2| - 1\} \geq I_d(z_1, z_2) = 0$. Hence, $I_d(z_1, z_2) \leq I_b(z_1, z_2)$.

**Corollary 3.2.5.** For all $z_1, z_2 \in \hat{z}$, $I_d(z_1, z_2) \leq I_b(z_1, z_2) \leq I_p(z_1, z_2) \leq I_m(z_1, z_2)$.
**Proof:** The proof follows from Proposition 3.2.1, Proposition 3.2.2, Proposition 3.2.3 and Proposition 3.2.4.

**Theorem 3.2.6**. If $I$ is an $L$-complex fuzzy intersection $t$-norms then $I(z_1, z_2 + z_3) = I(z_1, z_2) + I(z_1, z_3)$ if $I$ is an algebraic product that is $I = I_p$, $\forall\, z_1, z_2, z_3 \in \hat{z}$ and $|z_2 + z_3| \in L$.
**Proof**: By $t$-norms $I$ is an algebraic product we can say that

$$\begin{aligned} I(z_1, z_2 + z_3) &= |z_1| \cdot |z_2 + z_3| \\ &= |z_1 \cdot (z_2 + z_3)| = |z_1||z_2| + |z_1||z_3| \\ &= I(z_1, z_2) + I(z_1, z_3). \end{aligned}$$

**Theorem 3.2.7**. If $I$ is an algebraic product, then $I\big(I(z_1, z_2), I(z_3, z_4)\big) = I\big(I(z_1, z_3), I(z_2, z_4)\big)$ For all $z_1, z_2, z_3, z_4 \in \hat{z}$.
**Proof**: Since $I$ be an algebraic product so that

$$\begin{aligned} I\big(I(z_1, z_2), I(z_3, z_4)\big) &= |I(z_1, z_2)| \cdot |I(z_3, z_4)| = \big||z_1| \cdot |z_2|\big| \cdot \big||z_3| \cdot |z_4|\big| \\ &= \big||z_1| \cdot |z_2| \cdot |z_3| \cdot |z_4|\big| = \big||z_1| \cdot |z_3| \cdot |z_2| \cdot |z_4|\big| \\ &= \big||z_1| \cdot |z_3|\big| \cdot \big||z_2| \cdot |z_4|\big| = |I(z_1, z_3)| \cdot |I(z_2, z_4)| \\ &= I\big(I(z_1, z_3), I(z_2, z_4)\big). \end{aligned}$$

DOI: https://doi.org/10.31972/ticma2022

### 3.3 $L$-Complex Fuzzy Union $t$-Conorms

Let $\tilde{Z}_1$ and $\tilde{Z}_2$ be any two $L$-complex fuzzy sets with complex-valued membership functions $\mu(\tilde{Z}_1|x)$ and $\mu(\tilde{Z}_2|x)$, respectively. The union of $\tilde{Z}_1$ and $\tilde{Z}_2$ is defined by a function of the form $U: \hat{z}_1 \times \hat{z}_1 \longrightarrow \hat{z}_2$, which returns the membership function of $\tilde{Z}_1 \cup \tilde{Z}_1$ assigns $\mu(\tilde{Z}_1 \cup \tilde{Z}_1|x) = U(\mu(\tilde{Z}_1|x), \mu(\tilde{Z}_2|x))$, for all $x \in S$. The $L$-complex fuzzy union $U$ is a binary operation that satisfies at least four of the following properties for all $z_1$, $z_2$, $z_3$, $z_4 \in \hat{z}$:

(1) If $|z_2| = 0$, then $|U(z_1, z_2)| = |z_1|$;
(2) $|z_2| \leq |z_3|$ implies $|U(z_1, z_2)| \leq |U(z_1, z_3)|$;
(3) $U(z_1, z_2) = U(z_2, z_1)$;
(4) $U(z_1, U(z_2, z_4)) = U(U(z_1, z_2), z_4)$.

In most cases, it is desirable that $U$ should satisfy various additional requirements:
(5) $U$ is a continuous function;
(6) $|U(z_1, z_1)| > |z_1|$;
(7) $|z_1| \leq |z_3|$ and $|z_2| \leq |z_4| \Longrightarrow |U(z_1, z_2)| \leq |U(z_3, z_4)|$.

We have some examples that can be used for the $L$-complex fuzzy union as the following, each defined for all $z_1, z_2 \in \hat{z}$,
(1) $U_m(z_1, z_2) = \max(|z_1|, |z_2|)$.
(2) $U_P(z_1, z_2) = |z_1| + |z_2| - |z_1| \cdot |z_2|$.
(3) $U_b(z_1, z_2) = \min(|z_1| + |z_2|, 1)$.
(4) $U_d(z_1, z_2) = \begin{cases} |z_1| \; when \; |z_2| = 0 \\ |z_2| \; when \; |z_1| = 0. \\ 1 \quad otherwIse. \end{cases}$

**Proposition 3.3.1.** For all $z_1, z_2 \in \hat{z}$ then $U_m(z_1, z_2) \leq |U(z_1, z_2)| \leq U_d(z_1, z_2)$.
**Proof**: First we prove the left-hand right inequality. By the monotonicity condition, we have $|U(z_1, z_2)| \geq |U(z_1, 0)|$, and by boundary condition, we have to get $|U(z_1, z_2)| \geq |U(z_1, 0)| = |z_1|$. Next by commutatively, $|U(z_1, z_2)| = |U(z_2, z_1)| \geq |U(z_2, 0)| = |z_2|$. Hence, $|U(z_1, z_2)|$ is greater than or equal to $|z_1|$ and $|z_2|$, and this implies that $|U(z_1, z_2)| \geq max(|z_1|, |z_2|)$.
Now, we prove the second inequality, if $|z_2| = 0$ then by the boundary condition $|U(z_1, z_2)| = |z_1|$ and if $|z_1| = 0$ by the same condition, we get $|U(z_1, z_2)| = |z_2|$. Moreover, we have $|U(z_1, z_2)| \geq \max(|z_1|, |z_2|)$, so that $|U(z_1, 1)| \geq max(|z_1|, 1) = 1$ and $|U(1, z_1)| \geq max(1, |z_1|) = 1$. This is enough to say $|U(z_1, 1)| = |U(1, z_2)| = 1$. Therefore, by monotonicity condition, we get that $|U(z_1, z_2)| \leq |U(z_1, 1)| = |U(1, z_2)| = 1$.

**Proposition 3.3.2.** For all $z_1, z_2 \in \hat{z}$, then $U_b(z_1, z_2) \leq U_d(z_1, z_2)$.
**Proof:** First, if $|z_1| = 0$ (resp. $|z_2| = 0$), then $U_b(z_1, z_2) = U_d(z_1, z_2) = |z_1|$ (resp. $U_b(z_1, z_2) = U_d(z_1, z_2) = |z_2|$) respectively. Next, if $|z_1| = 1$ or $|z_2| = 1$, then $U_b(z_1, z_2) = U_d(z_1, z_2) = 1$. Finally, if $(|z_1|, |z_2| < k, k = supL)\backslash\{0,1\}$, then we say that $\min\{1, |z_1| + |z_2|\} \leq 1 = U_d(z_1, z_2)$. This implies that $U_b(z_1, z_2) \leq U_d(z_1, z_2)$.

**Proposition 3.3.3.** For all $z_1, z_2 \in \hat{z}$ then $U_m(z_1, z_2) \leq U_s(z_1, z_2)$.
**Proof:** At first, if $|z_1| = 1$ or $|z_2| = 1$, then $U_m(z_1, z_2) = U_s(z_1, z_2) = 1$. Next, if $|z_1| = 0$ (resp. $|z_2| = 0$), then $U_m(z_1, z_2) = U_s(z_1, z_2) = |z_2|$, (resp. $U_m(z_1, z_2) = U_s(z_1, z_2) = |z_1|$). Finally, if $(|z_1|, |z_2| < k, k = supL)\backslash\{0,1\}$, then we illustrate that $|z_1| + |z_2| - |z_1| \cdot |z_2| \geq |z_1|$, and $|z_1| + |z_2| - |z_1| \cdot |z_2| \geq |z_2|$, so that $|z_1| + |z_2| - |z_1| \cdot |z_2| \geq \max(|z_1|, |z_2|)$, implies that $U_m(z_1, z_2) \leq U_s(z_1, z_2)$.

**Proposition 3.3.4.** For all $z_1, z_2 \in \hat{z}$ then $U_s(z_1, z_2) \leq U_b(z_1, z_2)$.

**DOI: https://doi.org/10.31972/ticma2022**

**Proof:** In the case of $|z_1|$ or $|z_2|$ equals one or zero, $U_s(z_1, z_2) = U_b(z_1, z_2)$. But if $(|z_1|, |z_2| < k, k = supL)\backslash\{0,1\}$, then $U_s(z_1, z_2) = |z_1| + |z_2| - |z_1| \cdot |z_2| \leq U_b(z_1, z_2)$, because of $|z_1| + |z_2| - |z_1| \cdot |z_2| \leq |z_1| + |z_2|$.

**Corollary 3.3.5.** For all $z_1, z_2 \in \hat{z}$, $U_m(z_1, z_2) \leq U_s(z_1, z_2) \leq U_b(z_1, z_2) \leq U_d(z_1, z_2)$.
**Proof:** The proof follows from Proposition 3.3.1, Proposition 3.3.2, Proposition 3.3.3, and Proposition 3.3.4.

## 4. COMBINATIONS OF OPERATIONS

$L$-complex fuzzy sets satisfy the generalization of De Morgan's laws if and only if $|C(z)| = |z|$. That is

$$C\big(I(z_1, z_2)\big) = U\big(C(z_1), C(z_2)\big) \text{ and } C\big(U(z_1, z_2)\big) = I\big(C(z_1), C(z_2)\big)$$

for all $z, z_1, z_2 \in \hat{z}$.

**Theorem 4.1**. The operations $I_d$, $U_d$ and $L$-complex fuzzy complement are satisfied with the De Morgan's laws if $|C(z)| = |z|$, for all $z \in \hat{z}$.
**Proof**: First suppose $|z_1| = 0$, then by properties (1) of $C$, we have $|C(z_1)| = 1$, for all $z_1, z_2 \in \hat{z}$, so that we have two Cases,
Case (1); If $|z_2| = 1$, then by properties (1) of $C$ we get $|C(z_2)| = 0$, so that

$$C\big(U_d(z_1, z_2)\big) = C(|z_2|) = |C(z_2)| = I_d\big(C(z_1), C(z_2)\big)$$

and,

$$C\big(I_d(z_1, z_2)\big) = C(|z_1|) = |C(z_1)| = U_d\big(C(z_1), C(z_2)\big)$$

Case (2); If $|z_2|$ is not equal to one, implies that $|C(z_2)|$ not equal to zero, so that

$$C\big(I_d(z_1, z_2)\big) = C(0) = 1 = U_d\big(C(z_1), C(z_2)\big)$$

and,

$$C\big(U_d(z_1, z_2)\big) = C(|z_2|) = |C(z_2)| = I_d\big(C(z_1), C(z_2)\big)$$

Next, If $|z_2| = 0$, then $|C(z_2)| = 1$, for all $z_1, z_2 \in \hat{z}$, so that we have two cases
Case (i); If $|z_1| = 1$, then by properties (1) of $C$ we get that $|C(z_1)| = 0$, so that

$$C\big(U_d(z_1, z_2)\big) = C(|z_1|) = |C(z_1)| = I_d\big(C(z_1), C(z_2)\big)$$

and,

$$C\big(I_d(z_1, z_2)\big) = C(|z_2|) = |C(z_2)| = U_d\big(C(z_1), C(z_2)\big)$$

Case (ii) If $|z_1|$ is equal to one, implies that $|C(z_1)|$ is not equal to zero, so that

$$C\big(I_d(z_1, z_2)\big) = C(0) = 1 = U_d\big(C(z_1), C(z_2)\big)$$

and,

$$C\big(U_d(z_1, z_2)\big) = C(|z_1|) = |C(z_1)| = I_d\big(C(z_1), C(z_2)\big)$$

Finally, If $(|z_1|, |z_2| < k, k = supL)\backslash\{0,1\}$, then clear that $(C(z_1), C(z_2) < k, k = supL)\backslash\{0,1\}$.

$$C\left(I_p(z_1, z_2)\right) = C(0) = 1 = U_d\big(C(z_1), C(z_2)\big)$$

and,

$$C\big(U_d(z_1, z_2)\big) = C(1) = 0 = I_d\big(C(z_1), C(z_2)\big)$$

Hence, they satisfy De Morgan's laws.

**Theorem 4.2**. The operations $I_m$, $U_m$ and $L$-complex fuzzy complement $C$ are satisfies the De Morgan's laws if $|C(z)| = |z|$. For all $z \in \hat{z}$.
**Proof**: Suppose that $|z_1| \leq |z_2|$ for all $z_1, z_2 \in \hat{z}$, then by property (2) of $C$, clear that $|C(z_1)| \geq |C(z_2)|$, so that

$$\begin{aligned} C\big(I_m(z_1, z_2)\big) &= C(\min(|z_1|, |z_2|)) = C(|z_1|) = |C(z_1)|, \qquad by\ |C(z)| = |z| \\ &= \max\big((|C(z_1)|, |C(z_2)|)\big) = U_m\big(C(z_1), C(z_2)\big). \end{aligned}$$

and

$$C\big(U_m(z_1, z_2)\big) = C(\max(|z_1|, |z_2|)) = C(|z_2|) = |C(z_2)|, \qquad by\ |C(z)| = |z|$$

DOI: https://doi.org/10.31972/ticma2022

$$= \min\big((|C(z_1)|, |C(z_2)|)\big) = I_m\big(C(z_1), C(z_2)\big).$$

This implies they satisfy the De Morgan's laws.

**Theorem 4.3**. For a $t$-conorm $U$ and $L$-complex fuzzy complement $C$, the binary operation $I$ on $\hat{z}$ defined by $I(z_1, z_2) = C\left(U\big(C(z_1), C(z_2)\big)\right)$, is a $t$-norm if $|C(z)| = C|z|$, for all $z, z_1, z_2 \in \{z \mid z \in \mathbb{C}, |z| < k, k = supL\}$.

**Proof**: To prove that theorem we have to show that $I(z_1, z_2) = C\left(U\big(C(z_1), (z_2)\big)\right)$ is satisfied all conditions $I$.

(1) Let $|z_2| = 1$ then by properties (1) of $C$ we get $|C(z_2)| = 0$, so that by definition $I$ we have

$$\begin{aligned} |I(z_1, z_2)| &= \left|C\left(U\big(C(z_1), (z_2)\big)\right)\right| \\ &= C\big(\big|U\big(C(z_1), C(z_2)\big)\big|\big), \quad \text{by } |C(z)| = C|z|, \\ &= C(|C(z_1)|) = \big|C\big(C(z_1)\big)\big|, \text{by properties (2) } of\ I \text{ and } |C(z)| = C|z|, \\ &= |z_1|, \quad \text{by properties (4) of } C. \end{aligned}$$

Hence, $I$ satisfy property (1).

(2) for all $z_1, z_2, z_3 \in \hat{z}$. If $|z_2| \le |z_3|$ then by monotonicity of $C$, $|C(z_2)| \ge |C(z_3)|$. Moreover, by monotonicity of $U$

$$\big|U\big(C(z_1), C(z_2)\big)\big| \ge \big|U\big(C(z_1), C(z_3)\big)\big|,$$

Hence,

$$\begin{aligned} |I(z_1, z_2)| &= \big|C\big(U(C(z_1), |C(z_2)|)\big)\big| = C\big|\big(U(C(z_1), |C(z_2)|)\big)\big| \\ &\le C\big|\big(U(C(z_1), |C(z_3)|)\big)\big| = \left|C\left(U\big(C(z_1), C(z_3)\big)\right)\right| = |I(z_1, z_3)|. \end{aligned}$$

This implies that $I$ satisfies property (2).

(3) By commutativity of $U$ we have

$$I(z_1, z_2) = C\left(U\big(C(z_1), C(z_2)\big)\right) = C\left(U\big(C(z_2), C(z_1)\big)\right) = I(z_1, z_2)$$

So that, $I$ satisfies property (3).

(4) For any $z_1, z_2, z_3 \in \hat{z}$. Then

$$\begin{aligned} I\big(z_1, I(z_2, z_3)\big) &= C\left(U\left(C(z_1), C\big(I(z_2, z_3)\big)\right)\right) \\ &= C\left(U\left(C(z_1), C\left(C\left(U\big(C(z_2), C(z_3)\big)\right)\right)\right)\right) \\ &= C\left(U\left(C(z_1), \left(U\big(C(z_2), C(z_3)\big)\right)\right)\right), \text{by property (4)of } C \\ &= C\left(U\left(U\big(C(z_1), C(z_2)\big), C(z_3)\right)\right), \text{by property (4)of } U \\ &= C\left(U\left(C\left(C\left(U\big(C(z_1), C(z_2)\big)\right)\right), C(z_3)\right)\right), \text{by property (4)of } C \\ &= I\big(I(z_1, z_2), z_3\big) \end{aligned}$$

Hence, $I$ satisfys property (4), so that it is a $L$-complex fuzzy $t$-norm.

**Theorem 4.4.** $L$-complex fuzzy $t$-norm $I(z_1, z_2) = C\left(U\big(C(z_1), C(z_2)\big)\right)$ is satisfies the De morgan's laws, for any $z_1, z_2 \in \hat{z}$.

**Proof**: By definition $I$ we have

$$\begin{aligned} C\big(I(z_1, z_2)\big) &= C\left(C\left(U\big(C(z_1), C(z_2)\big)\right)\right) \\ &= U\big(C(z_1), C(z_2)\big), \text{by property (4) of } C. \end{aligned}$$

$$I(C(z_1), C(z_2)) = C\left(U\left(C(C(z_1)), C(C(z_2))\right)\right)$$
$$= C(U(z_1, z_2)), \text{By property (4) of } C.$$

**Theorem 4.5**. For a $t$-norm $I$ and an $L$-complex fuzzy complement $C$, the binary operation $U$ on $\hat{z}$ defined by $U(z_1, z_2) = C\left(I(C(z_1), C(z_2))\right)$, is $t$-conorm if $|C(z)| = C|z|$, for all $z, z_1, z_2 \in \hat{z}$.
**Proof:** Similar steps with Theorem 4.3.
Hence, $U$ satisfies properties (4), so that it is $L$-complex fuzzy $t$-conorm.
**Theorem 4.6.** $L$-complex fuzzy $t$-conorm $U(z_1, z_2) = C(I(C(z_1), |C(z_2)|))$ is satisfies the De morgan laws, for any $z_1, z_2 \in \hat{z}$.
**Proof:** Analogous to the proof of Theorem 4.4.

## 5. CONCLUSIONS:

This work introduced a new form of set, the $L$-complex fuzzy set, where $L$ is a completely distributive lattice. This is an extension of the notion of a complex fuzzy set. This work also proposed various applications for the idea of an $L$-complex fuzzy set. The research started with a look at the basic set-theoretic operations of complement, union and intersection, as well as how these apply to the complicated $L$-complex fuzzy set. On $L$-complex fuzzy sets, basic operations and properties were developed derived.

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