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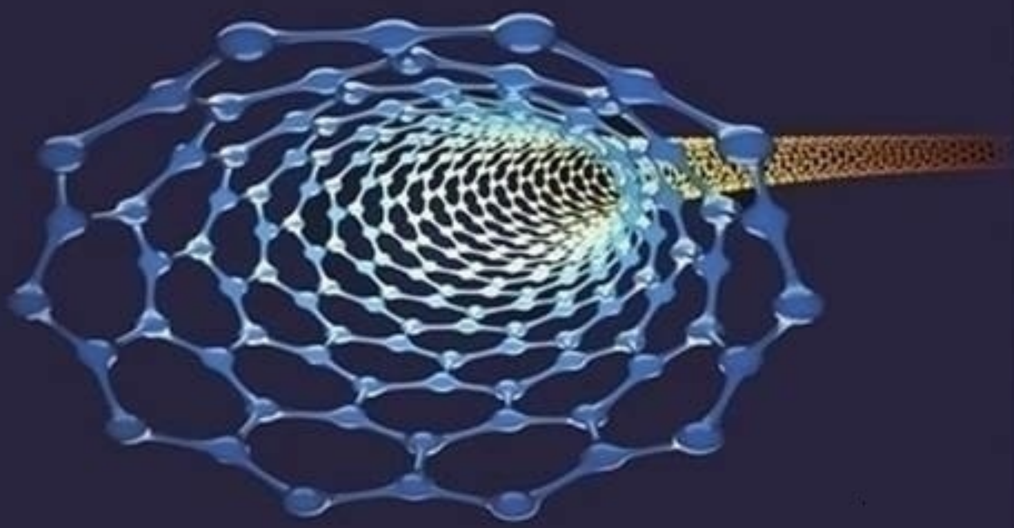

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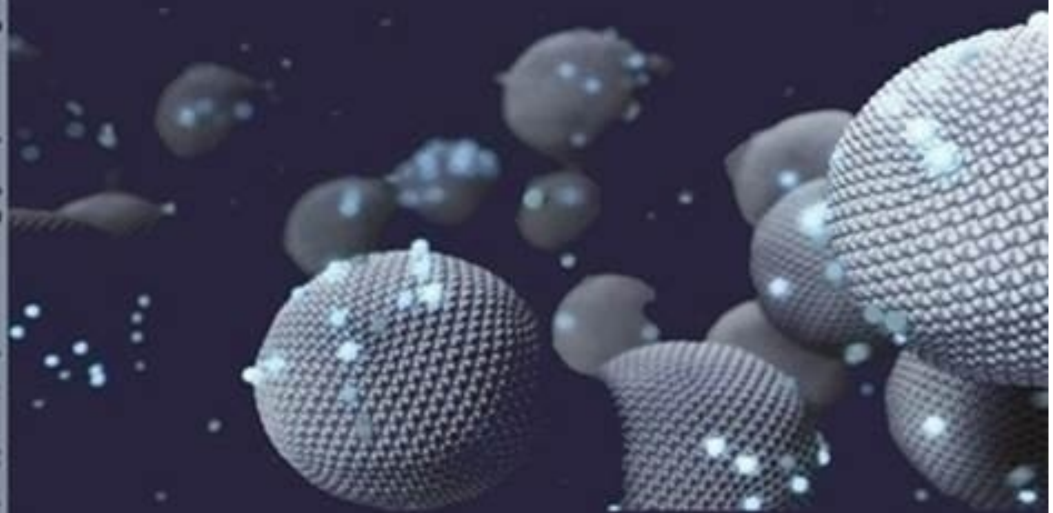

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RESEARCH PAPER

ω_p-Open and ω_p-Closed Functions

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A B S T R A C T: In this *work*, we study and define two new concepts of functions named ω_p- *open* and ω_p- *closed* functions by using the concepts of $\omega_p - open$ and $\omega_p - closed$ sets. The concept of $\omega_p - open$ function strictly located between both the concepts of *open* and *preopen* functions. We obtain a few properties of these functions, however, the connections between them are examined.

KEYWORDS: ω_p-*closed* set, ω_p-*open* set, ω_p-*continuous function*, $\omega_p - closed$ function, ω_p-open function.
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1. INTRODUCTION:

In 1963, Levine [2] defined a new class of *open* sets called semi-*open* sets, also he introduced a new class of functions named *semi-continuous* and semi-*open* functions in the space of topology. Mashhour et al. [3] presented *pre-continuous*, *weak pre-continuous* and *pre-open* functions. The concepts of α–*continuous* and α-*open* functions are investigated and defined by Mashhour et al. [4].

Abd El-Monsef et el. [5] represented a new class of sets called $\beta - open$ sets, and they described β-*continuous* and β-*open* functions. The notion of the γ-*open* function is investigated by El-Atik [6]. However, Raychaudhuri and Mukherjee [7] defined δ–*preopen* sets, also they present δ–*almost continuous* and δ-*preopen* functions.

The purpose of the *paper* is, we apply the notions of ω_p- *open* and ω_p- *closed* set to describe the new types of functions denoted by $\omega_p - open$ and $\omega_p - closed$ functions. In addition, the basic properties and the relation between these functions are presented.

2. PRELIMINARIES

All through the present paper (X, τ) and $(Y, \Im)$ express the *spaces* of *topology* on which no *separation* axioms are considered otherwise is clarified, also *Fun* means function. If $\mathcal{D} \subseteq$ X, then the *interior* (resp. $\omega - interior$, $\delta - interior$, $\omega_p - interior$) of $\mathcal{D}$ is the union of all *open* (resp. $\omega - open$, $\delta - open$, $\omega p - open$) sets in X contained in $\mathcal{D}$ represented by $Int(\mathcal{D})$ (resp. $\omega Int(\mathcal{D})$, $Int_\delta(\mathcal{D})$, $\omega_p Int(D)$. The *closure* (resp. $\omega - closure$, $\delta - closure$, $\omega_p - closure$) of $\mathcal{D}$ is the *intersection* of all *closed* (resp. $\omega - closed$, $\delta - closed$, $\omega_p - closed$) sets of X containing $\mathcal{D}$. A subset $\mathcal{D}$ in X is called $semi - open$ [2] (resp. $regular - open$ [8], *preopen* [3], α–*open* [9], $\beta - open$, [5], $\gamma - open$ [6], δ–*preopen* [7], $\omega_p - open$ [1]) if $\mathcal{D} \subseteq Cl(Int(\mathcal{D}))$ (resp. $\mathcal{D} = Int(Cl(\mathcal{D}))$, $\mathcal{D} \subseteq Int(Cl(\mathcal{D}))$, $\mathcal{D} \subseteq Int(Cl(Int(\mathcal{D})))$, $\mathcal{D} \subseteq Cl(Int(Cl(\mathcal{D})))$, $\mathcal{D} \subseteq Int(Cl(\mathcal{D})) \cup Cl(Int(\mathcal{D}))$, $G \subseteq Int(Cl_\delta(\mathcal{D}))$, $\mathcal{D} \subseteq Int(\omega Cl(\mathcal{D}))$).

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Also, a subset $\mathcal{D}$ of X is called $\delta - open$ [10] if $\mathcal{D}$ is the $union$ of all $regular - open$ subset of X, and a subset $\mathcal{D}$ is called $\omega - open$ [11] if for $each\ x \in A$, $there\ exits$ an $open$ set $\mathcal{U}$ in X containing x such that $\mathcal{U}- D$ is $countable$. The family of all $semi - open$ (resp. $regular - open$, $preopen$, $\alpha - open$, $\beta - open$, $\gamma - open$, $\delta - open$, $\delta - open$, $\omega - open$, $\omega_p - open$) subsets of X represented by $SO(X)$ (resp. $RO(X)$, $PO(X)$, $\alpha\text{-}O(X)$, $\beta\text{-}O(X)$, $\gamma\text{-}O(X)$, $\delta\text{-}O(X)$, $\delta\text{-}PO(X)$, $\omega O(X)$, $\omega_p O(X)$). The complement of $semi - open$ (resp. $regular - open$, $preopen$, $\alpha - open$, $\beta - open$, $\gamma - open$, $\delta - preopen$, $\omega - open$, $\omega_p - open$) set is called $semi - closed$ (resp. $regular - closed$, $preclosed$, $\alpha - closed$, $\beta - closed$, $\gamma - closed$, $\delta - preclosed$, $\omega - closed$ $\omega_p - closed$), also their family is represented by $SC(X)$ (resp. $RC(X)$, $PC(X)$, $\alpha - C(X)$, $\beta - C(X)$, $\gamma - C(X)$, $\delta - C(X)$, $\delta - PC(X)$, $\omega C(X)$, $\omega_p C(X)$).

Definition 2.1. Let $\mathfrak{h}: X \to Y$ be a Fun. If $\mathfrak{h}(\mathcal{D})$ is $open$ (resp. $semi - open$, $preopen$, $\alpha - open$, $\beta - open$, $\delta - preopen$, $\gamma - open$) in Y, each $open$ subset $\mathcal{D}$ of X, thus, $\mathfrak{h}$ is called $open$ (resp. $semi - open$ [12], $preopen$ [3], $\alpha - open$ [4], $\beta - open$ [5], $\gamma - open$ [6], $\delta - preopen$ [7]) Fun.

Definition 2.2. Let $\mathfrak{h}: X \to Y$ be a Fun. If $\mathfrak{h}^{-1}(\mathcal{D})$ is $open$ ($\omega_p - open$) in X, each $open$ subset $\mathcal{D}$ of Y, thus, $\mathfrak{h}$ is called $continuous$ [13] ($\omega_p - continuous$ [1]) Fun.

Definition 2.3. ([14]) Let (X, τ) be a $space$ of $topology$. Then, a $space$ X is said to be:

1. $Locally\ countable$, if each $point$ $x \in X$ has a $countable$ $open\ neighborhood$.
2. $Submaximal$, if every $preopen$ set is $open$, equivalently if every dense subset of X is $open$ in X.

Lemma 2.4. ([1]) For a set $\mathcal{D}$ in space X, the $followings$ are true:

1. Let $\mathcal{D}$ be an $open$ set. Then it is $\omega_p - open$.
2. Let $\mathcal{D}$ be an $\omega_p - open$. Then it is $pre - open$, $pre - \omega - open$ and $\delta - preopen$.

Theorem 2.5. ([16]) Let (X, τ) be a $locally\ countable\ space$. Then, $\tau^{\omega} = \tau_{dis}$.

Proposition 2.6. ([1]) A subset $\mathcal{D}$ of $space$ X is $\omega_p - open$ ($\omega_p - closed$) $\Leftrightarrow \omega_p Int(\mathcal{D}) = \mathcal{D}$ ($\omega_p Cl(\mathcal{D}) = \mathcal{D}$).

3. MORE PROPERTIES OF ω_p-OPEN SETS

Theorem 3.1. Let (X, τ) be a $locally\ countable\ space$. Then, a set $\mathcal{D}$ in X is $\omega_p - open \Leftrightarrow$ its $open$.
Proof: If $\mathcal{D}$ is an $\omega_p - open$ subset of a $locally\ countable\ space$ X, so $\omega Cl(\mathcal{D}) = \mathcal{D}$, by Theorem 2.5, so $A \subseteq Int(\omega Cl(\mathcal{D})) = Int(\mathcal{D})$. This means that, $\mathcal{D}$ is an $open$ set. Conversely, let $\mathcal{D}$ be an $open$ set in X. Then, $\mathcal{D} = Int(\mathcal{D})$, so $\mathcal{D} \subseteq Int(\omega Cl(\mathcal{D}))$. Hence, $\mathcal{D}\ is$ an $\omega_p - open$ set.

Theorem 3.2. If (X, τ) is a $submaximal\ space$, then a set $\mathcal{D}$ of X is $\omega_p - open \Leftrightarrow$ its $preopen$.
Proof: Let $\mathcal{D}$ be an $\omega_p - open$ set in X. Then, by Lemma 2.4, $\mathcal{D}$ is $preopen$. Conversely, assume $\mathcal{D}$ is a $preopen$ set in $a\ submaximal\ space\ X$. Then, $\mathcal{D}$ is $open$. By part (1) of Lemma 2.4, $\mathcal{D}$ is $\omega_p - open$.

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4.ω_p-Open Functions

Definition 4.1. A Fun $\mathfrak{h}: X \longrightarrow Y$ is $said\ to$ be $\omega_p - open$, if the $image$ of $each\ open$ set in X is $\omega_p - open$ in Y.

Theorem 4.2. Let $\mathfrak{h}: X \longrightarrow Y$ be a Fun. Then, $\mathfrak{h}$ is an $\omega_p - open \Leftrightarrow$ for $each$ $x \in X$ and $each\ open$ set $\mathcal{U}$ in X containing x, there $exists\ an\ \omega_p - open\ set\ \mathcal{V}$ in Y containing $\mathfrak{h}(x)$ such that $\mathcal{V} \subseteq \mathfrak{f}(\mathcal{U})$.

Proof: Suppose $\mathcal{U}$ is an $open$ set in X such that $x \in \mathcal{U}$. Then, $\mathfrak{h}(\mathcal{U})$ is $\omega_p - open\ in\ Y$, and $\mathfrak{h}(x) \in \mathfrak{f}\,(\mathcal{U})$. Put $\mathcal{V} = \mathfrak{h}(\mathcal{U})$ is $\omega_p - open$, $\mathfrak{h}(x) \in \mathcal{V}$ and $\mathcal{V} = \mathfrak{h}(\mathcal{U})$. Conversely, let $\mathcal{U}$ be an $open\ set\ in\ X$. To show $\mathfrak{h}$ is $\omega_p - open$. We must show $\mathfrak{h}(\mathcal{U})$ is $\omega_p - open$. If $\mathfrak{h}(\mathcal{U}) = \emptyset$, then its $\omega_p - open$. Otherwise, let $y \in \mathfrak{h}(\mathcal{U})$. Then, there $exists$ $x \in \mathcal{U}$ such that $\mathfrak{h}(x) = y$. Since $x \in \mathcal{U}$, so by hypothesis, there $exists$ an $\omega_p - open$ subset $\mathcal{V}$ of X such that $\mathfrak{h}(x) \in \mathcal{V} \subseteq \mathfrak{h}(\mathcal{U})$ that is, $y \in \mathcal{V} \subseteq \mathfrak{h}(\mathcal{U})$. Therefore, $y \in \omega_p Int(\ \mathfrak{h}(\mathcal{U}))$. This implies that, $\mathfrak{h}(\mathcal{U}) = \omega_p Int(\ \mathfrak{h}(\mathcal{U}))$ which by Proposition 2.6, means $\mathfrak{h}(\mathcal{U})$ is $an\ \omega_p - open$ set in Y. Hence $\mathfrak{h}$ is $\omega_p - open$.

Theorem 4.3. The f$ollowing$ conditions are $equivalent$ for that Fun $\mathfrak{h}$ from a space X to space Y:

1. $\mathfrak{h}$ is an $\omega p{-}open$;
2. $\mathfrak{h}\,(Int(\mathcal{D})) \subseteq \omega_p Int(\mathfrak{h}(\mathcal{D}))$, for $each$ $\mathcal{D} \subseteq X$;
3. $Int(\mathfrak{h}^{-1}(\mathcal{D})) \subseteq \mathfrak{h}^{-1}(\omega_p Int(\mathcal{D}))$, for $each$ $\mathcal{D} \subseteq Y$;
4. $\mathfrak{h}^{-1}(\omega_p Cl(\mathcal{D})) \subseteq Cl(\mathfrak{h}^{-1}(\mathcal{D}))$, for $each$ $\mathcal{D} \subseteq Y$.

Proof: (1) $\Rightarrow$ (2) Let $\mathcal{D} \subseteq X$. Then, $Int(\mathcal{D})$ is $open$ in X. By (1), $\mathfrak{h}\,(Int(\mathcal{D}))$ is $\omega_p - open$ in Y, implies that,

$$\omega_p Int(\mathfrak{h}\,(Int(\mathcal{D}))) = \mathfrak{h}\,(Int(\mathcal{D})) \quad (*)$$

$Since$, $\mathfrak{h}\,(Int(\mathcal{D})) \subseteq \mathfrak{h}\,(\mathcal{D})$, then, $\omega_p Int(\mathfrak{h}\,(Int(\mathcal{D}))) \subseteq \omega_p Int(\mathfrak{h}(\mathcal{D}))$, thus by $(*)$, $\mathfrak{h}\,(Int(\mathcal{D})) \subseteq \omega_p Int(\mathfrak{h}\,(\mathcal{D}))$.

(2) $\Rightarrow$ (3) Suppose $\mathcal{U}\ is\ any\ subset\ of\ Y$. Then, $\mathfrak{h}^{-1}(\mathcal{U}) \subseteq X$, so by (2), $\mathfrak{h}\,(Int(\mathfrak{h}^{-1}(\mathcal{U}))) \subseteq \omega_p Int(\mathfrak{h}\,(^{-1}(\mathcal{U}))) \subseteq \omega_p Int(\mathcal{U})$. Therefore, $Int(\mathfrak{h}^{-1}(\mathcal{U})) \subseteq \mathfrak{h}^{-1}(\omega_p Int(\mathcal{U}))$.

(3) $\Rightarrow$ (4) Let $\mathcal{U} \subseteq Y$. Then, $Y - \mathcal{U} \subseteq Y$. Thus by (3), $Int(\mathfrak{h}^{-1}(Y - \mathcal{U})) \subseteq \mathfrak{h}^{-1}(\omega_p Int(Y - \mathcal{U}))$. So, $X - Cl(\mathfrak{h}^{-1}(\mathcal{U})) \subseteq X - \mathfrak{h}^{-1}(\omega_p Cl(\mathcal{U}))$. That is, $\mathfrak{h}^{-1}(\omega_p Cl(\mathcal{U})) \subseteq Cl(\mathfrak{h}^{-1}(\mathcal{U}))$.

(4) $\Rightarrow$ (1) Let $\mathcal{D}$ be an $open$ subset of X. Then, by (3), $Int(\mathfrak{h}^{-1}(\mathfrak{h}\,(\mathcal{D}))) \subseteq \mathfrak{h}^{-1}(\omega_p Int(\mathfrak{h}\,(\mathcal{D})))$. Since $Int(\mathfrak{h}\,(\mathfrak{h}^{-1}(\mathcal{D}))) \subseteq \mathfrak{h}^{-1}(\omega_p Int(\mathfrak{h}\,(\mathcal{D})))$, so $Int(\mathcal{D}) \subseteq Int(\mathfrak{h}^{-1}(\mathfrak{h}\,(\mathcal{D})))$, implies that $Int(\mathcal{D}) \subseteq \mathfrak{h}^{-1}(\omega_p Int(\mathfrak{h}(\mathcal{D})))$, but since $\mathcal{D}$ is $open$, then $\mathcal{D} \subseteq \mathfrak{h}^{-1}(\omega_p Int(\mathfrak{h}\,(\mathcal{D})))$. Thus, $\mathfrak{h}(\mathcal{D}) \subseteq \mathfrak{h}\,(\mathfrak{h}^{-1}(\omega_p Int(\mathfrak{h}(\mathcal{D})))) \subseteq \omega_p Int(\mathfrak{h}(\mathcal{D}))$. Therefore, $\mathfrak{h}(\mathcal{D})$ is $\omega_p{-}open$. Hence, $\mathfrak{h}$ an is $\omega_p - open\ Fun$.

Remark 4. 4. The $following$ implications show the $relationship\ between\ \omega_p - open\ Fun$ and other types of $open\ Fun$.

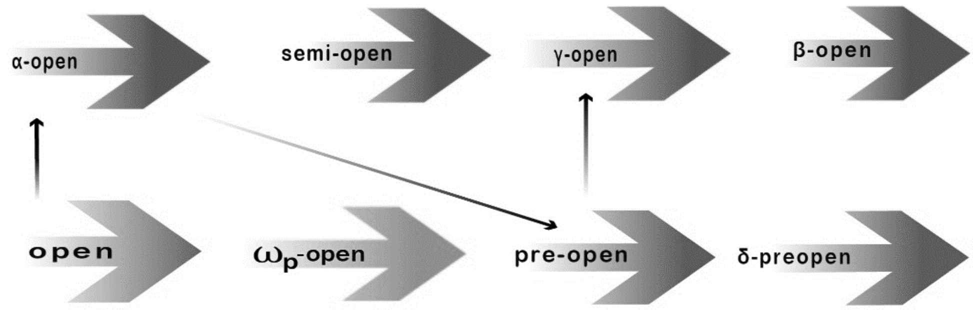

FIGURE 1. $shows\ the\ relationship\ between\ \omega_p - open\ Fun\ and\ other\ types\ of\ open\ Fun$

The $following$ examples show that the converse of the implications in Figure 1 is not $true$ in $general$.

Example 4.5. If $X = Y = \{1,2,3\}$, and $\tau = \Im = \{\emptyset, X, \{1\}, \{1,2\}\}$, then the Fun $\mathfrak{f} : (X, \tau) \to (Y, \rho)$ is $defined$ by, $\mathfrak{f}(1) = 1, \mathfrak{f}(2) = 3\ and\ \mathfrak{f}(3) = 2$ is $preopen$, but not $\omega_p - open$. Since, $PO(Y) = \{\emptyset, Y, \{1\}, \{1,2\}, \{1,3\}\}$ and $\omega_p O(Y) = \{\emptyset, Y, \{1\}, \{1,2\}\}$. Thus, $\mathfrak{f}(\{1,2\}) = \{1,3\}$ is not $\omega_p - open$ in Y.

Example 4.6. Consider $X = \mathbb{R}$ with the $usual\ topology\ \Im_u$ and $Y = \mathbb{R}$ $with$ the $co - countable\ topology$ τ_{coc}. The $identity\ Fun\ \iota : (X, \Im_u) \to (Y, \tau_{coc})$ is $\omega_p - open\ but$ not $open$. Since $(0,1)$ is $open$ in $(\mathbb{R}, \Im_u)$, but $\iota((0,1)) = (0,1)$ is not $(\mathbb{R}, \tau_{coc})$, so it is not $open$ while for any $open$ subset $\mathcal{D}$ of $(\mathbb{R}, \Im_u)$, there is an $open$ interval (a, b) subset of $\mathcal{D}$, and $\iota(\mathcal{D}) = \mathcal{D} \subseteq \mathbb{R} = Int(Cl((a,b))) = Int(\omega Cl((a,b))) \subseteq Int(\omega Cl(\mathcal{D})) = Int(\omega Cl(\iota(\mathcal{D})))$ which is $\omega_p - open$ in $(\mathbb{R}, \tau_{coc})$, hence its $\omega_p - open\ Fun$.

Theorem 4.7. If $\zeta : X \to Y$ is an $open\ Fun$ and $\eta : Y \to Z$ is an $\omega_p - open\ Fun$, then $\eta \circ \zeta$ is an $\omega_p - open$ Fun.

Proof: Let $\mathcal{V}$ be an $open$ subset of X. Then, by $hypothes$is, $\zeta(\mathcal{V})$ is $open$ in Y. Since η is $\omega_p - open$, thus $\eta(\zeta(\mathcal{V}))$ is $\omega_p - open$ in Z. Therefore, $\eta \circ \zeta$ is $\omega_p - open$.

Proposition 4.8. Let $\mathfrak{h}$ be a Fun from any space (X, τ) to a $locally\ countable\ space$ (Y, ρ). Then, $\mathfrak{h}$ is $\omega_p - open \Leftrightarrow$ its $open$.

Proof: Let $\mathfrak{h}$ be an $\omega_p - open\ Fun$. Then, for $each\ open$ set $\mathcal{U}$ in X, thus $\mathfrak{h}(\mathcal{U})$ is $\omega_p - open$ in Y. Since, Y is $locally\ countable$, thus by $Theorem$ 3.1, $\mathfrak{h}(\mathcal{U})$ $open$ in Y. Therefore, $\mathfrak{h}$ is $open$. Conversely, it follows $from\ Lemma$ 2.4.

Proposition 4.9. Let $\mathfrak{h}$ be a Fun from any $space$ (X, τ) to a $submaximal\ space$ (Y, ρ). Then, $\mathfrak{h}$ is $preopen$ $\Leftrightarrow$ its $\omega_p - open$.

Proof: It follows from $Proposition$ 3.2.

Theorem 4.10. For the $Funs$ $\mathfrak{g} : X \to Y$ and $\mathfrak{h} : Y \to Z$, the $following\ conditions$ are $true$:

1. If $\mathfrak{h} \circ \mathfrak{g}$ is an $open\ Fun$ and $\mathfrak{h}$ is an $injective\ \omega_p - continuous\ Fun$, then $\mathfrak{g}$ is $\omega_p - open$.
2. If $\mathfrak{h} \circ \mathfrak{g}$ is an $\omega_p - open\ Fun$ and $\mathfrak{g}$ is an $surjective$ $continuous\ Fun$, then $\mathfrak{h}$ is $\omega_p - open$.

Proof:

1. Let $\mathcal{U}$ be any $open$ subset of X. Then, by $hypothesis$, $\mathfrak{h}(\mathfrak{g}(\mathcal{U}))$ is $open$ in Z. Since $\mathfrak{h}$ is $injective$ $\omega_p - continuous$, thus, $\mathfrak{h}^{-1}\left(\mathfrak{h}(\mathfrak{g}(\mathcal{U}))\right) = \mathfrak{g}(\mathcal{U})$ is $\omega_p - open$ in Y. Hence, $\mathfrak{g}$ is $\omega_p - open$.

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2. Suppose $\mathcal{U}$ is an $open$ subset of Y. By $hypothesis$, $\mathfrak{g}^{-1}(\mathcal{U})$ is $open$ in X. Since, $\mathfrak{h}\circ\mathfrak{g}$ is $\omega_p - open$, $\mathfrak{h}\circ\mathfrak{g}(\mathfrak{g}^{-1}(\mathcal{U}))$ is $\omega_p - open$ in Z. Since, $\mathfrak{g}$ is $surjective$, so $\mathfrak{h}(\mathcal{U}) = \mathfrak{h}\,(\mathfrak{g}(\mathfrak{g}^{-1}(\mathcal{U})))$. Hence, $\mathfrak{h}$ is an $\omega_p - open\ Fun$.

Theorem 4.11. Let $\mathfrak{h} : X \to Y$ be an $\omega_p - open\ Fun$. If $\mathcal{K}$ is an $open\ subspace$ of X, then the $restriction$ $Fun\ \mathfrak{h}_{/\mathcal{K}} : \mathcal{K} \to Y$ is $\omega_p - open$.

Proof: Let $\mathcal{D}$ be any $open$ subset of $\mathcal{K}$. Since $\mathcal{K}$ is $open$ in X, implies that $\mathcal{D}$ is $open$ in X. By $hypothesis$, $\mathfrak{h}(\mathcal{D})$ is $\omega_p - open$ in Y. But, $\mathfrak{h}_{/\mathcal{K}}(\mathcal{D}) = \mathfrak{h}(\mathcal{D})$. Therefore, $\mathfrak{h}_{/\mathcal{K}}$ is $\omega_p - open$ an Fun.

5. ω_p-CLOSED FUNCTIONS

Definition 5.1. A $Fun\ \mathfrak{h} : X \to Y$ is $said$ to be an $\omega_p - closed\ Fun$, if the $image$ of $each\ closed$ set in X is $\omega_p - closed$ in Y.

Theorem 5.2. Let $\mathfrak{h} : X \to Y$ be a Fun. Then, $\mathfrak{h}$ is $\omega_p - closed \Leftrightarrow \omega_p Cl(\mathfrak{h}(\mathcal{D})) \subseteq \mathfrak{h}(Cl(\mathcal{D}))$, for $each$ $\mathcal{D} \subseteq X$.
Proof: Let $\mathfrak{h}$ be an $\omega_p - closed\ Fun$ and $\mathcal{D} \subseteq X$. Then, $Cl(\mathcal{D})$ is a $closed$ subset of X. By $hypothesis$, $\mathfrak{h}(Cl(\mathcal{D}))$ is $\omega_p - closed$ in Y. That is, $\omega_p Cl(\mathfrak{h}(Cl(\mathcal{D}))) = \mathfrak{h}(Cl(\mathcal{D}))$. Since, $\mathfrak{h}(\mathcal{D}) \subseteq \mathfrak{h}(Cl(\mathcal{D}))$, then $\omega_p Cl(\mathfrak{h}(\mathcal{D})) \subseteq \omega_p Cl(\mathfrak{h}(Cl(\mathcal{D}))) = \mathfrak{h}(Cl(\mathcal{D}))$. Therefore, $\omega_p Cl(\mathfrak{h}(\mathcal{D})) \subseteq \mathfrak{h}(Cl(\mathcal{D}))$. Conversely, we must show that $\mathfrak{h}$ is $\omega_p - closed$. Let $\mathcal{D}$ be a $closed$ subset of X. By $hypothesis$, $\omega_p Cl(\mathfrak{h}(\mathcal{D})) \subseteq \mathfrak{h}(Cl(\mathcal{D})) = \mathfrak{h}(\mathcal{D})$. That is, $\mathfrak{h}(\mathcal{D})$ is $\omega_p - closed$ in Y.

Theorem 5.3. The $following\ statement$ are $equivalent$, for a $bijective\ Fun\ \mathfrak{h} : X \to Y$:
1. $\mathfrak{h}$ is an $\omega_p - closed$;
2. $\mathfrak{h}^{-1}(\omega_p Cl(\mathcal{D})) \subseteq Cl(\mathfrak{h}(\mathcal{D}))$, for $each\ \mathcal{D} \subseteq Y$;
3. $Int(\mathfrak{h}^{-1}(\mathcal{D})) \subseteq \mathfrak{h}^{-1}(\omega_p Int(\mathcal{D}))$, for $each\ \mathcal{D} \subseteq Y$.

Proof:
(1) $\Rightarrow$ (2) Let $\mathfrak{h}$ be a $\omega_p - closed$ and $\mathcal{D} \subseteq Y$. Then, $Cl(\mathfrak{h}(\mathcal{D}))$ is $closed$ in X. By $hypothesis$, $\mathfrak{h}(Cl(\mathfrak{h}^{-1}(\mathcal{D})))$ is $\omega_p - closed$ in Y, this implies that,

$$\omega_p\, Cl\left(\mathfrak{h}\left(Cl(\mathfrak{h}^{-1}(\mathcal{D}))\right)\right) = \mathfrak{h}\left(Cl(\mathfrak{h}^{-1}(\mathcal{D}))\right) \qquad (**)$$

Since $\mathfrak{h}$ is $bijective$, thus $\mathcal{D} = \mathfrak{h}(\mathfrak{h}^{-1}(\mathcal{D})) \subseteq \mathfrak{h}(Cl(\mathfrak{h}^{-1}(\mathcal{D})))$. By $(**)$ $\omega_p Cl(\mathcal{D}) \subseteq \omega_p Cl(\mathfrak{h}(Cl(\mathfrak{h}^{-1}(\mathcal{D})))) = \mathfrak{h}(Cl(\mathfrak{h}^{-1}(\mathcal{D})))$. Therefore, $\mathfrak{h}^{-1}(\omega_p Cl(\mathcal{D})) \subseteq Cl(\mathfrak{h}^{-1}(\mathcal{D}))$.

(2) $\Rightarrow$ (3) Let $\mathcal{U} \subseteq Y$. Then, $Y - \mathcal{U} \subseteq Y$, by (2), $\mathfrak{h}^{-1}(\omega_p Cl(Y - \mathcal{U})) \subseteq Cl(\mathfrak{h}^{-1}(Y - \mathcal{U}))$. Thus, $X - \mathfrak{h}^{-1}(\omega_p Int(\mathcal{U})) \subseteq X - Int(\mathfrak{h}^{-1}(\mathcal{U}))$, this means that $Int(\mathfrak{h}^{-1}(\mathcal{U})) \subseteq \mathfrak{h}^{-1}(\omega_p Int(\mathcal{U}))$.

(3) $\Rightarrow$ (1) Let $\mathcal{D}$ be a $closed$ subset of X. To show that, $\mathfrak{h}$ is $\omega_p - closed$. Since, $\mathfrak{h}(X - \mathcal{D}) \subseteq Y$, so by (3), $Int((\mathfrak{h}(X - \mathcal{D}))) \subseteq \mathfrak{h}^{-1}(\omega_p Int(\mathfrak{h}(X - \mathcal{D}))$. By $bijective$ of $\mathfrak{h}$, $X - Cl(\mathcal{D}) \subseteq X - \mathfrak{h}^{-1}(\omega_p Cl(\mathfrak{h}(\mathcal{D})))$, then $\omega_p Cl(\mathfrak{h}(\mathcal{D})) \subseteq (\mathfrak{h}(Cl(\mathcal{D})))$. But $\mathcal{D}$ is $closed$, thus $\omega_p Cl(\mathfrak{h}(\mathcal{D})) \subseteq \mathfrak{h}(\mathcal{D})$, and $\mathfrak{h}\,(\mathcal{D})$ is $closed$ in Y. Hence, $\mathfrak{h}$ $is\ \omega_p - closed$.

Theorem 5.4. Let $\mathfrak{h}: X \to Y$ be an $\omega_p - closed\ Fun$. If $\mathcal{K}$ is a $closed\ subspace$ in X, then the $restriction$
$Fun\ \mathfrak{h}_{/\mathcal{K}} : \mathcal{K} \to Y$ is $\omega_p - closed$.
Proof: Let $\mathcal{D}$ be any $closed$ subset of $\mathcal{K}$. Since $\mathcal{K}$ is $closed$ in X, implies that $\mathcal{D}$ is $closed$ in X.
$By\ hyp$othesis,

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$\mathfrak{h}(\mathcal{D})$ is $\omega_p -$ *closed* in Y. But, $\mathfrak{h}_{/\mathcal{K}}(\mathcal{D}) = \mathfrak{h}(\mathcal{D})$. Therefore, $\mathfrak{h}_{/\mathcal{K}}$ is $\omega_p -$ *closed Fun*.

Theorem 5.5. Let $\zeta: X \rightarrow Y$ be a *closed* Fun and $\eta: Y \rightarrow Z$ be an $\omega_p -$ *closed Fun*. Then, $\eta \circ \zeta$ is an $\omega_p -$ *closed Fun*.
Proof: Let $\mathcal{D}$ be any *closed* subset of X. Then, by *hypothesis*, $\zeta(\mathcal{D})$ is *closed* in Y. Since η is $\omega_p -$ *closed*, thus $\eta(\zeta(\mathcal{D}))$ is *an* $\omega_p -$ *closed set* in Z. Therefore, $\eta \circ \zeta$ is $\omega_p -$ *closed*.

Theorem 5.6. The *following conditions* are *true*, for the *Funs* $\mathfrak{g}: X \rightarrow Y$ and $\mathfrak{h}: Y \rightarrow Z$:

1. If $\mathfrak{h} \circ \mathfrak{g}$ is a *closed Fun* and $\mathfrak{h}$ is an *injective* $\omega_p -$ *continuous Fun*, then $\mathfrak{g}$ is $\omega_p -$ *closed*.
2. If $\mathfrak{h} \circ \mathfrak{g}$ is an $\omega_p -$ *closed Fun* and $\mathfrak{g}$ is an *surjective* continuous *Fun*, then $\mathfrak{h}$ is $\omega_p -$ *closed*.

Proof.

1. Let $\mathcal{D}$ be any *closed* subset of X. Then, by *hypothesis*, $\mathfrak{h}(\mathfrak{g}(\mathcal{D}))$ is *closed* in Z. Since $\mathfrak{h}$ is *injective*

$\omega_p -$ *continuous*, $\mathfrak{h}^{-1}\left(\mathfrak{h}\left(\mathfrak{g}(\mathcal{D})\right)\right) = \mathfrak{g}(\mathcal{D})$ is $\omega_p -$ *closed* in Y. Hence, $\mathfrak{g}$ *is* $\omega_p -$ *open*.

2. Suppose $\mathcal{D}$ is a *closed* subset of Y. Since $\mathfrak{g}$ is *continuous*, then, $\mathfrak{g}^{-1}(\mathcal{D})$ is *closed* in X. By *hyp*othesis,

$\mathfrak{h}\left(\mathfrak{g}\left(\mathfrak{g}^{-1}(\mathcal{D})\right)\right) = \mathfrak{h}(\mathcal{D})$ is *an* $\omega_p -$ *closed* set in Z. Therefore, $\mathfrak{h}$ is $\omega_p -$ *closed*.

Theorem 5.7. The *following* conditions are *equivalent*, for a *bijective Fun* $\mathfrak{h}: X \rightarrow Y$:

1. h is $\omega_p -$ *continuous*;
2. h is $\omega_p -$ *open*;
3. h is $\omega_p -$ *closed*.

Proof:

(1) $\Rightarrow$ (2) Let $\mathcal{D}$ be any *open* set in X. Since $\mathfrak{h}^{-1}$ is $\omega_p -$ *continuous*, thus, $(\mathfrak{h}^{-1})^{-1}(\mathcal{D}) = \mathfrak{h}(\mathcal{D})$ is $\omega_p -$ *ope*
in Y. Therefore, $\mathfrak{h}$ is $\omega_p -$ *open*.

(2) $\Rightarrow$ (3) Let $\mathfrak{h}$ be an $\omega_p -$ *open Fun*, and $\mathcal{D}$ be a *closed* set in X. Then, $X - \mathcal{D}$ is *open*. By *hypothesis*, $\mathfrak{h}(X - \mathcal{D}) = Y - \mathfrak{h}(\mathcal{D})$ is $\omega_p -$ *open* in Y. Thus, $\mathfrak{h}(\mathcal{D})$ is $\omega_p -$ *closed*.

(3) $\Rightarrow$ (1) Let $\mathcal{D}$ be any *open* set in X. Then, $X - \mathcal{D}$ is *closed*. Since $\mathfrak{h}$ is *bijective* and by (3), $\mathfrak{h}(X - \mathcal{D}) =$
$Y - \mathfrak{h}(\mathcal{D})$ is $\omega_p -$ *closed* in Y. Thus, $(\mathfrak{h}^{-1})^{-1}(\mathcal{D}) = \mathfrak{h}(\mathcal{D})$ is $\omega_p -$ *open*. Hence, $\mathfrak{h}^{-1}$ is $\omega_p -$ *open*.

6. CONCLUSION:

In this *paper*, we showed that every $\omega_p -$ *open* set in *locally countable* space is *open*. We used the *notions* of $\omega_p -$ *open* and $\omega_p -$ *closed* sets to describe $\omega_p -$ *open* and $\omega_p -$ *closed Fun*. Also, there is a relationship between *open* and *preopen Funs* via $\omega_p -$ *open Funs*. We acquired some *fundamental theorems* and properties of these *Funs*. However, there are combined $\omega_p -$ *continuous Funs* with $\omega_p -$ *open* and $\omega_p -$ *closed Fun*.

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